**Page 7**

**1. Addition**

To add two complex numbers, you add their real parts together and their imaginary parts together. The sum of $(a+bi)$ and $(c+di)$ is:

$$(a+c)+(b+d)i$$

Here's the step-by-step process:

- Add the real parts: $a+c$
- Add the imaginary parts: $b+d$
- Combine them into the form of a complex number: real part + imaginary part $\cdot i$

**2. Multiplication**

To multiply two complex numbers, you use the distributive property (FOIL: First, Outer, Inner, Last), just as you would with binomials. Here's the process:

$$\begin{aligned}(a+bi)(c+di) &= ac+adi+bci+bidi\\ &= ac+adi+bci-bd \;\; (\text{since } i^2=-1)\\ &= (ac-bd)+(ad+bc)i\end{aligned}$$

The product of $(a+bi)$ and $(c+di)$ is a new complex number whose real part is $(ac-bd)$ and whose imaginary part is $(ad+bc)$.

**3. Division**

To divide two complex numbers, you multiply the numerator and denominator by the complex conjugate of the denominator. The complex conjugate of $c+di$ is $c-di$. This gets rid of the imaginary part in the denominator.

The division is performed as follows:

$$\begin{aligned}\frac{a+bi}{c+di}\cdot\frac{c-di}{c-di} &= \frac{(a+bi)(c-di)}{c^2+d^2}\\ &= \frac{(ac+bd)+(bc-ad)i}{c^2+d^2}\end{aligned}$$

The complex conjugate was chosen because $(c+di)(c-di)=c^2-(di)^2=c^2-(-d^2)=c^2+d^2$, which is a real number, thus eliminating the imaginary unit from the denominator.

The result is:

$$\frac{(ac+bd)}{c^2+d^2}+\frac{(bc-ad)}{c^2+d^2}i$$

**Page 16**

Let $\mathbf{x}$ be an $m$-dimensional vector and $\mathbf{1}_n$ be an $n$-dimensional row vector of ones. The matrix $A \in \mathbb{R}^{m\times n}$ represented as the outer product of $\mathbf{x}$ and $\mathbf{1}_n$ is given by:

$$A = \mathbf{x}\mathbf{1}_n^T = \begin{bmatrix} x_1 & x_1 & \cdots & x_1 \\ x_2 & x_2 & \cdots & x_2 \\ \vdots & \vdots & \ddots & \vdots \\ x_m & x_m & \cdots & x_m \end{bmatrix}$$

**Page 22**

- Addition $+$ and multiplication $\times$ are binary operations on $\mathbb{N}$.
- Subtraction $-$ and division $/$ are not always binary operations on $\mathbb{N}$.

**Page 28**

To determine if $B$ under these operations is a field, we must check the following properties:

- **Closure** is satisfied for both addition and multiplication since all operations result in either 0 or 1.
- **Associativity** is satisfied for both operations in a set with two elements since there are not enough elements to construct a counterexample.
- **Commutativity** is shown in the tables as the tables are symmetric about the diagonal.
- **Additive Identity**: 0 is the additive identity since $0 + 0 = 0$ and $0 + 1 = 1$.
- **Multiplicative Identity**: 1 is the multiplicative identity since $1 \times 0 = 0$ and $1 \times 1 = 1$.
- **Additive Inverse**: Each element is its own additive inverse since $0+0 = 0$ and $1+1 = 0$.
- **Multiplicative Inverse**: The element 1 is its own multiplicative inverse. The element 0 does not need a multiplicative inverse since it is the additive identity.
- **Distributive Law**: To check this, we would need to verify that $a\times(b+c) = (a\times b)+(a\times c)$ for all $a$, $b$, and $c$ in $B$, but this is evident from the tables provided.

Given that all these properties are satisfied by the operations as defined in the tables, $B = \{0, 1\}$ under these operations does indeed form a field, specifically the finite field of order 2, often denoted by $\mathbb{F}_2$.

**Page 29**

1. $\mathbb{R}$ - The set of real numbers is a field. It satisfies all the properties of a field: closure, associativity, commutativity, and distributivity for both addition and multiplication; there are additive and multiplicative identities (0 and 1, respectively); and every element has an additive inverse, and every non-zero element has a multiplicative inverse.
2. $\mathbb{C}$ - The set of complex numbers is a field, with all the same properties as the real numbers, extended to include the imaginary unit $i$, where $i^2 = -1$.
3. $\mathbb{Q}$ - The set of rational numbers (fractions of integers, excluding division by zero) is a field.
4. $\mathbb{Z}$ - The set of integers is not a field because integers do not have multiplicative inverses within the integers (for example, there is no integer $x$ such that $2x = 1$).
5. $\mathbb{W}$ - The set of whole numbers (non-negative integers, including zero) is not a field because, like the integers, they lack multiplicative inverses for any number other than 1 and 0 (and 0 does not have a multiplicative inverse).
6. $\mathbb{N}$ - The set of natural numbers (positive integers) is not a field for the same reasons as the whole numbers and integers: there are no multiplicative inverses within the set for numbers other than 1.
7. $\mathbb{R}^{2\times 2}$ - This denotes the set of $2 \times 2$ matrices with real number entries. This set is not a field because not every non-zero element (matrix) has a multiplicative inverse. For instance, matrices that have a determinant of zero do not have an inverse.