به نام خداوند بخشنده مهربان

جبر خطی

دانشکده مهندسی کامپیوتر

حمیدرضا ربیعی، مریم رمضانی
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Example

Suppose $\varphi, \tau \in V'$. Then the bilinear form α on V defined by

$$\alpha(u, \omega) = \varphi(u)\tau(\omega) - \varphi(\omega)\tau(u)$$

is alternating.

To show that α is alternating، we need to verify that $\alpha(u, u) = 0$ for all $u \in V$.

$$\alpha(u, u) = \varphi(u)\tau(u) - \varphi(u)\tau(u)$$
$$\alpha(u, u) = \varphi(u)\tau(u) - \varphi(u)\tau(u) = 0$$

Thus،

$$\alpha(u, u) = 0 \text{ all for } u \in V$$

Therefore، the bilinear form α is alternating.

Theorem

A bilinear form α on V is alternating if and only if

$$\alpha(u,\omega) = -\alpha(\omega,u)$$

for all $u,\omega \in V$.

Proof First suppose that α is alternating. If $u, w \in V$, then

$$\begin{aligned} 0 &= \alpha(u + w, u + w) \\ &= \alpha(u, u) + \alpha(u, w) + \alpha(w, u) + \alpha(w, w) \\ &= \alpha(u, w) + \alpha(w, u). \end{aligned}$$

Thus $\alpha(u, w) = -\alpha(w, u)$, as desired.

To prove the implication in the other direction, suppose $\alpha(u, w) = -\alpha(w, u)$ for all $u, w \in V$. Then $\alpha(v, v) = -\alpha(v, v)$ for all $v \in V$, which implies that $\alpha(v, v) = 0$ for all $v \in V$. Thus α is alternating. ■

Theorem

The sets $V^{(2)}_{\text{sym}}$ and $V^{(2)}_{\text{alt}}$ are subspaces of $V^{(2)}$. Furthermore,

$$V^{(2)} = V^{(2)}_{\text{sym}} \oplus V^{(2)}_{\text{alt}}$$

Proof The definition of symmetric bilinear form implies that the sum of any two symmetric bilinear forms on V is a bilinear form on V, and any scalar multiple of any bilinear form on V is a bilinear form on V. Thus $V^{(2)}_{\text{sym}}$ is a subspace of $V^{(2)}$. Similarly, the verification that $V^{(2)}_{\text{alt}}$ is a subspace of $V^{(2)}$ is straightforward.

Next, we want to show that $V^{(2)} = V^{(2)}_{\text{sym}} + V^{(2)}_{\text{alt}}$. To do this, suppose $\beta \in V^{(2)}$. Define $\rho, \alpha \in V^{(2)}$ by

$$\rho(u, w) = \frac{\beta(u, w) + \beta(w, u)}{2} \quad \text{and} \quad \alpha(u, w) = \frac{\beta(u, w) - \beta(w, u)}{2}.$$

Then $\rho \in V^{(2)}_{\text{sym}}$ and $\alpha \in V^{(2)}_{\text{alt}}$, and $\beta = \rho + \alpha$. Thus $V^{(2)} = V^{(2)}_{\text{sym}} + V^{(2)}_{\text{alt}}$.

Finally, to show that the intersection of the two subspaces under consideration equals $\{0\}$, suppose $\beta \in V^{(2)}_{\text{sym}} \cap V^{(2)}_{\text{alt}}$. Then (*) implies that

$$\beta(u, w) = -\beta(w, u) = -\beta(u, w)$$

for all $u, w \in V$, which implies that $\beta = 0$. Thus $V^{(2)} = V^{(2)}_{\text{sym}} \oplus V^{(2)}_{\text{alt}}$, as implied by (**) ■

(*) با استفاده از قضیه قبلی

(**) قضیه $Direct sum$

Example

Suppose $\alpha, \rho \in V^{(2)}$. Define a function $\beta : V^4 \to F$ by Then $\beta \in V^4$

$$\beta(v_1, v_2, v_3, v_4) = \alpha(v_1, v_2)\rho(v_3, v_4).$$

We need to show that β can be expressed as a sum of simpler terms، each involving only one input، to demonstrate its linearity with respect to each input. This is done using the superposition property.

For input $v_1 = (x_1, y_1)$، the superposition property states:

$$\beta(av_1 + b, v_2, v_3, v_4) = \alpha(av_1 + b, v_2)\rho(v_3, v_4)$$

Expanding the expression using the definition of β:

$$= \alpha(av_1 + b, v_2) \cdot \rho(v_3, v_4)$$

Since $v_1 = (x_1, y_1)$، we can rewrite $av_1 + b$ as $a(x_1, y_1) + b(x_1, y_1) = (ax_1 + b, ay_1 + b)$.

Applying the function α to this composite input:

$$= \alpha(ax_1 + b, ay_1 + b, x_2, y_2) \cdot \rho(v_3, v_4)$$

Now، we can use linearity of α with respect to each argument:

$$= a \cdot \alpha(x_1, y_1, x_2, y_2) + b \cdot \alpha(x_1, y_1, x_2, y_2) \cdot \rho(v_3, v_4)$$

This simplifies to:

$$= a\beta(v_1, v_2, v_3, v_4) + b\beta(v_1, v_2, v_3, v_4)$$

This demonstrates linearity with respect to v_1.

The rest is similar.

Theorem

$V_{\text{alt}}^{(m)}$ is a subspace of $V^{(m)}$.

Proof that $V_{\textbf{alt}}^{(m)}$ is a Subspace of $V^{(m)}$

To prove that $V_{\text{alt}}^{(m)}$ is a subspace of $V^{(m)}$, we need to show that $V_{\text{alt}}^{(m)}$ satisfies the following three criteria:

(a) The Zero m-Linear Form

The zero m-linear form $0 \in V_{\text{alt}}^{(m)}$ since:

$$0(v_{\sigma(1)}, \ldots, v_{\sigma(m)}) = 0 = \operatorname{sgn}(\sigma) \cdot 0(v_1, \ldots, v_m).$$

Thus, $0 \in V_{\text{alt}}^{(m)}$.
Another way: The zero m-linear form α_0 defined by $\alpha_0(v_1, v_2, \ldots, v_m) = 0$ for all $v_1, v_2, \ldots, v_m \in V$ is clearly alternating because it trivially satisfies $\alpha_0(v_1, \ldots, v_j, \ldots, v_k, \ldots, v_m) = 0$ whenever $v_j = v_k$. Thus, $\alpha_0 \in V_{\text{alt}}^{(m)}$.

(b) Closure under Addition

Let $\alpha, \beta \in V_{\text{alt}}^{(m)}$. We need to show that $\alpha + \beta \in V_{\text{alt}}^{(m)}$:

$$(\alpha + \beta)(v_{\sigma(1)}, \ldots, v_{\sigma(m)}) = \operatorname{sgn}(\sigma) \cdot (\alpha + \beta)(v_1, \ldots, v_m).$$

Since both α and β are alternating, their sum $\alpha + \beta$ is also alternating. Thus, $\alpha + \beta \in V_{\text{alt}}^{(m)}$.
Another way:

$$(\alpha + \beta)(v_1, v_2, \ldots, v_m) = \alpha(v_1, v_2, \ldots, v_m) + \beta(v_1, v_2, \ldots, v_m) = 0 + 0 = 0$$

Hence, $\alpha + \beta$ is also an alternating m-linear form, implying that $\alpha + \beta \in V_{\text{alt}}^{(m)}$.

(c) Closure under Scalar Multiplication

Let $\alpha \in V_{\text{alt}}^{(m)}$ and c be a scalar. We need to show that $c\alpha \in V_{\text{alt}}^{(m)}$:

$$(c\alpha)(v_{\sigma(1)}, \ldots, v_{\sigma(m)}) = \operatorname{sgn}(\sigma) \cdot (c\alpha)(v_1, \ldots, v_m).$$

For any scalar c, if α is alternating, then $c\alpha$ will also be alternating. Thus, $c\alpha \in V_{\text{alt}}^{(m)}$.
Another way:

$$(c\alpha)(v_1, v_2, \ldots, v_m) = c \cdot \alpha(v_1, v_2, \ldots, v_m) = c \cdot 0 = 0$$

Hence, $c\alpha$ is also an alternating m-linear form, implying that $c\alpha \in V_{\text{alt}}^{(m)}$.

Theorem

Suppose m is a positive integer and α is an alternating m-linear form on V. If $v_1, \dots, v_m$ is a linearly dependent list in V, then

$$\alpha(v_1, \dots, v_m) = 0$$

Proof Suppose $v_1, \dots, v_m$ is a linearly dependent list in V. By the linear dependence lemma (*) , some v_k is a linear combination of $v_1, \dots, v_{k-1}$. Thus there exist $b_1, \dots, b_{k-1}$ such that $v_k = b_1 v_1 + \cdots + b_{k-1} v_{k-1}$. Now

$$\begin{aligned} \alpha(v_1, \dots, v_m) &= \alpha\Big(v_1, \dots, v_{k-1}, \sum_{j=1}^{k-1} b_j v_j, v_{k+1}, \dots, v_m\Big) \\ &= \sum_{j=1}^{k-1} b_j \, \alpha(v_1, \dots, v_{k-1}, v_j, v_{k+1}, \dots, v_m) \\ &= 0. \end{aligned}$$

■

(*) قضیه عنوان شده در اسلاید صفحه‌ی ۱۸

Theorem

فرض کنید m (تعداد بردارها) $> dim\ V$. در نتیجه ۰ تنها $alternating\ m-linear\ form$ به روی V است.

فرض کنید α یک $alternating\ m-linear\ form$ روی V باشد و $v_1, ..., v_m \in V$. از آن جایی که $m > dim\ V$ این لیست مستقل خطی نیست. پس به این نتیجه می رسیم که $\alpha\ (v_1, ..., v_m) = 0$. پس α یک تابع صفر از V^m به F است.

Theorem

فرض کنید m یک عدد طبیعی، α یک $alternating\ m-linear\ form$ روی V و $v_1, ..., v_m$ لیستی از بردارها در V باشند. در نتیجه‌ی جابه‌جایی جایگاه هر دو بردار در $\alpha\ (v_1, ..., v_m)$ مقدار α با یک فاکتور -1 تغییر می‌کند.

نخست $v_1 + v_2$ را در دو جایگاه نخست قرار می‌دهیم و به عبارت زیر می‌رسیم.

$$\alpha(v_1 + v_2, v_1 + v_2, v_3, ..., v_m) = 0$$

اکنون از $multilinear$ بودن α برای گسترش سمت چپ معادله استفاده می‌کنیم تا به عبارت زیر برسیم.

$$\alpha(v_2, v_1, v_3, ..., v_m) = -\alpha(v_1, v_2, v_3, ..., v_m)$$

به طور مشابه می‌توان این عمل را برای بردارهای جایگاه‌های دیگر نیز انجام داد و به نتیجه‌ی یکسان رسید.

Theorem

اگر m یک عدد طبیعی باشد و $\alpha \in V_{alt}^{(m)}$ ، برای هر لیست $v_1, ..., v_m$ از بردارهای درون V و هر $(j_1, ..., j_m) \in perm(m)$ داریم
$$\alpha(v_{j_1}, ..., v_{j_m}) = sign(j_1, .., j_m)\ \alpha(v_1, ..., v_m)$$

با توجه به این که $j_1, ..., j_m$ جایگشتی از ۱ تا m هستند، می‌توانیم با تعدادی جابه‌جایی از این جایگشت، به جایگشت $1, .., m$ برسیم. هر بار این جابه‌جایی‌ها اندازه‌ی α را با فاکتور -1 تغییر می‌دهد و در نتیجه مقدار باقی جایگشت را نیز با همین فاکتور تغییر می‌دهد. پس از تعداد مشخصی جابه‌جایی به جایگشت $1, ..., m$ می‌رسیم که در آن $sign = 1$. پس اگر مقدار α به تعداد بار زوج تغییر کرد یعنی $sign(j_1, ..., j_m) = 1$ و اگر به تعداد بار فرد تغییر کرد یعنی $sign(j_1, ..., j_m) = -1$ که نتیجه‌ی مورد نظر را می‌دهد.

Theorem

فرض کنید $n = dim\ V$. همچنین $e_1, ..., e_n$ یک مجموعه‌ی پایه برای V هستند و $v_1, ..., v_n \in V$. برای هر $k \in 1, ..., n$ اگر داشته باشیم $b_{1,k}, ..., b_{n,k} \in F$ به شکلی که $v_k = \sum_{j=1}^{n} b_{j,k} e_j$ می‌توان نشان داد که برای هر α که یک $alternating\ n-linear\ form$ روی V است،

$$\alpha(v_1, ..., v_n) = \alpha(e_1, ..., e_n)\ \sum_{(j_1,...,j_n) \in perm(n)} (sign(j_1, ..., j_n))\ b_{j_1,1}, ..., b_{j_n,n}$$

اگر α یک $alternating\ n-linear\ form$ روی V باشد، داریم

$$\alpha(v_1, ..., v_n) = \alpha(\sum_{j_1=1}^{n} b_{j_1,1} e_{j_1}, ..., \sum_{j_n=1}^{n} b_{j_n,n} e_{j_n}) =$$
$$\sum_{j_1=1}^{n} ... \sum_{j_n=1}^{n} b_{j_1,1} ... b_{j_n,n}\ \alpha(e_{j_1}, ..., e_{j_n}) = \sum_{(j_1,...,j_n) \in perm(n)} b_{j_1,1} ... b_{j_n,n}\ \alpha(e_{j_1}, ..., e_{j_n}) =$$
$$\alpha(e_1, ..., e_n)\ \sum_{(j_1,...,j_n) \in perm(n)} (sign(j_1, ..., j_n))\ b_{j_1,1}, ..., b_{j_n,n}$$

معادله‌ی سوم به این دلیل برقرار است که $\alpha(e_{j_1}, ..., e_{j_n}) = 0$ اگر $j_1, ..., j_n$ متمایز نباشند.

Theorem

بُعد فضای برداری $V_{alt}^{(dim\ V)}$ برابر یک است.

فرض کنید $n = dim\ V$ و α و α' دو $alternating\ n-linear\ form$ روی V هستند که $\alpha \neq 0$. اگر $e_1,...,e_n$ به گونه‌ای باشند که $\alpha(e_1,...,e_n) \neq 0$ ، وجود دارد $c \in \mathbf{F}$ به شکلی که $\alpha'(e_1,...,e_n) = c\alpha(e_1,...,e_n)$. اکنون طبق قضیه‌ی استقلال خطی در $alternating\ n-linear form$ می‌دانیم که $e_1,...,e_n$ مستقل خطی و در نتیجه یک پایه برای V هستند.
اکنون فرض کنید $v_1,...,v_n \in V$ و $b_{j,k}$ درست مانند قضیه‌ی پیش برای هر $j,\ k\ =\ 1,...,n$ انتخاب شده باشند. در نتیجه

$$\begin{aligned}\alpha'(v_1,...,v_n) &= \alpha'(e_1,...,e_n)\ \textstyle\sum_{(j_1,...,j_n)\in perm(n)} (sign(j_1,...,j_n))\, b_{j_1,1},...,b_{j_n,n}\\ &= c\alpha(e_1,...,e_n)\ \textstyle\sum_{(j_1,...,j_n)\in perm(n)} (sign(j_1,...,j_n))\, b_{j_1,1},...,b_{j_n,n}\\ &= c\alpha(v_1,...,v_n)\end{aligned}$$

رابطه‌ی بالا از قضیه‌ی پیش به دست می‌آید. معادله‌ی بالا نشان می‌دهد که $\alpha' = c\alpha$ در نتیجه $\alpha,\ \alpha'$ یک لیست مستقل خطی نیست که نتیجه می‌دهد $dim\ V_{alt}^{(n)} \leq 1$.
برای پایان اثبات نیاز است نشان دهیم که یک $alternating\ n-linear\ form$ ناصفر روی V وجود دارد و در نتیجه گزینه‌ی بُعد صفر از میان می‌رود. به این منظور مجدد فرض می‌کنیم $e_1,...,e_n$ یک پایه برای V و $\phi_1,...,\phi_n \in V'$ توابع خطی روی V باشند که به ما اجازه می‌دهند هر عنصر از V را به عنوان ترکیب خطی از این پایه نشان دهیم. به بیان دیگر برای هر $v \in V$ داریم

$$v = \sum_{j=1}^{n} \phi_j(v)e_j$$

اکنون برای $v_1,...,v_n \in V$ ، α را به این شکل تعریف می‌کنیم.

$$\alpha(v_1,...,v_n) = \sum_{(j_1,...,j_n)\in perm(n)} (sign(j_1,...,j_n))\ \phi_{j_1}(v_1),...,\phi_{j_n}(v_n)$$

این که α یک $n-linear\ form$ است، مشهود می‌باشد. برای اثبات $alternating$ بودن α فرض کنید $v_1 = v_2$. برای هر $(j_1,...,j_n) \in perm(n)$ جایگشت $(j_2,j_1,j_3,...,j_n)$ علامت مخالف دارد. از آن جایی که $v_1 = v_2$ تاثیر این دو جایگشت در جمع درون رابطه حنثی می‌شود و نتیجه می‌گیریم $\alpha(v_1,v_1,v_3,...,v_n) = 0$. به شکل مشابه اگر هر دو برداری در لیست $v_1,...,v_n$ برابر باشند، $\alpha(v_1,...,v_n) = 0$. این نشان می‌دهد که α یک $alternating\ n-linear\ form$ است.
در نهایت نیز به سادگی می‌توان نشان داد $\alpha(e_1,...,e_n) = 1 \neq 0$ پس بُعد فضای برداری یک است.

Theorem

Suppose that n is a positve integer. The map that takes a list $v_1, ..., v_n$ of vectors in F^n to det($v_1, ..., v_n$ is an alternating n-linear form of F^n.

Let $e_1, ..., e_2$ be the standard basis of F^n and suppose $v_1, ..., v_n$ is a list of vectors in F^n. Let $T \in \mathcal{L}(F^n)$ be the operator such that $Te_k = v_k$ for $k = 1, ..., n$. Thus T is the operator whose matrix with respect to $e_1, ..., e_n$ is $(v_1 ... v_n)$. Hence det$(v_1 ... V_n)$ = det$T$، by definition of the determinant of a matrix.

Let α be alternating n-lineaar form on F^n such that $\alpha(e_1, ..., e_n) = 1$. Then

$$\begin{aligned} \det(v_1 ... v_n) &= \det T \\ &= (\det T)\alpha(e_1, ..., e_n) \\ &= \alpha(Te_1, ..., Te_n) \\ &= \alpha(v_1, ..., v_n), \end{aligned}$$

where the third line follows from the definition of the determinant of an operator. The equation above shows that the map that takes a list of vectors $v_1, ..., v_n$ in F^n to $\det(v_1 ... v_n)$ is the alternating n-linear form α on F^n.

Theorem

Suppose that n is a positive integer and A is an n-by-n square matrix. Then

$$\det A = \sum_{(j_1,...,j_n)\in perm(n)} (sign(j_1,...,j_n))A_{j_1,1}...A_{j_n,n}$$

Theorem * : Let $n = \dim V$. Suppose $e_1,...,e_n$ is a basis of V and $v_1,...,v_n \in V$. For each $k \in \{1,...,n\}$، let $b_{1,k},...,b_{n,k} \in F$ be such that

$$v_k = \sum_{j=1}^{n} b_{j,k}e_j$$

$$Then$$

$$\alpha(v_1,...,v_n) = \alpha(e_1,...,e_n) \sum_{(j_1,...,j_n)\in perm(n)} (sign(j_1,...,j_n))b_{j_1,1}...b_{j_n,n}$$

for every alternating n-linear form α on V.
Apply theorem * with $V = F^n$ and $e_1,...,e_n$ the standard basis of F^n and α the alternating n-linear form on F^n that takes $v_1,...,v_n$ to $\det(v_1...v_n)$. If each v_k is the k^{th} column of A، then each $b_{j,k}$ in theorem * equals $A_{j,k}$. Finally،

$$\alpha(e_1,...,e_n) = \det(e_1,,,e_n) = \det I = 1.$$

Example

- Determinant of 2 * 2 matrix
- Determinant of 3 * 3 matrix

$$\det\begin{bmatrix}a & c\\ b & d\end{bmatrix}:$$

$$V(\begin{bmatrix}a & c\\ b & d\end{bmatrix}) = V(\begin{bmatrix}a\\ b\end{bmatrix},\begin{bmatrix}c\\ d\end{bmatrix}) = V(a\begin{bmatrix}1\\ 0\end{bmatrix}+b\begin{bmatrix}0\\ 1\end{bmatrix},c\begin{bmatrix}1\\ 0\end{bmatrix}+d\begin{bmatrix}0\\ 1\end{bmatrix})$$

$$= aV(\begin{bmatrix}1\\ 0\end{bmatrix},c\begin{bmatrix}1\\ 0\end{bmatrix}+d\begin{bmatrix}0\\ 1\end{bmatrix}) + bV(\begin{bmatrix}0\\ 1\end{bmatrix},c\begin{bmatrix}1\\ 0\end{bmatrix}+d\begin{bmatrix}0\\ 1\end{bmatrix})$$

$$= acV(\begin{bmatrix}1\\ 0\end{bmatrix},\begin{bmatrix}1\\ 0\end{bmatrix}) + adV(\begin{bmatrix}1\\ 0\end{bmatrix},\begin{bmatrix}0\\ 1\end{bmatrix}) + bcV(\begin{bmatrix}0\\ 1\end{bmatrix},\begin{bmatrix}1\\ 0\end{bmatrix}) + bdV(\begin{bmatrix}0\\ 1\end{bmatrix},\begin{bmatrix}0\\ 1\end{bmatrix})$$

$$= 0 + adV(\begin{bmatrix}1 & 0\\ 0 & 1\end{bmatrix}) + bcV(\begin{bmatrix}0 & 1\\ 1 & 0\end{bmatrix}) + 0 = ad - bcV(\begin{bmatrix}1 & 0\\ 0 & 1\end{bmatrix}) = ad - bc$$

$$\det\begin{bmatrix}a & b & c\\ d & e & f\\ g & h & i\end{bmatrix}:$$

$$V(\begin{bmatrix}a & b & c\\ d & e & f\\ g & h & i\end{bmatrix}) = V(\begin{bmatrix}a\\ d\\ g\end{bmatrix},\begin{bmatrix}b\\ e\\ h\end{bmatrix},\begin{bmatrix}c\\ f\\ i\end{bmatrix}) = V(a\begin{bmatrix}1\\ 0\\ 0\end{bmatrix}+d\begin{bmatrix}0\\ 1\\ 0\end{bmatrix}+g\begin{bmatrix}0\\ 0\\ 1\end{bmatrix},b\begin{bmatrix}1\\ 0\\ 0\end{bmatrix}+e\begin{bmatrix}0\\ 1\\ 0\end{bmatrix}+h\begin{bmatrix}0\\ 0\\ 1\end{bmatrix},c\begin{bmatrix}1\\ 0\\ 0\end{bmatrix}+f\begin{bmatrix}0\\ 1\\ 0\end{bmatrix}+i\begin{bmatrix}0\\ 0\\ 1\end{bmatrix})$$

$$= aV(\begin{bmatrix}1\\ 0\\ 0\end{bmatrix},b\begin{bmatrix}1\\ 0\\ 0\end{bmatrix}+e\begin{bmatrix}0\\ 1\\ 0\end{bmatrix}+h\begin{bmatrix}0\\ 0\\ 1\end{bmatrix},c\begin{bmatrix}1\\ 0\\ 0\end{bmatrix}+f\begin{bmatrix}0\\ 1\\ 0\end{bmatrix}+i\begin{bmatrix}0\\ 0\\ 1\end{bmatrix}) + dV(\begin{bmatrix}0\\ 1\\ 0\end{bmatrix},b\begin{bmatrix}1\\ 0\\ 0\end{bmatrix}+e\begin{bmatrix}0\\ 1\\ 0\end{bmatrix}+h\begin{bmatrix}0\\ 0\\ 1\end{bmatrix},c\begin{bmatrix}1\\ 0\\ 0\end{bmatrix}+f\begin{bmatrix}0\\ 1\\ 0\end{bmatrix}+i\begin{bmatrix}0\\ 0\\ 1\end{bmatrix})$$

$$+gV(\begin{bmatrix}0\\ 0\\ 1\end{bmatrix},b\begin{bmatrix}1\\ 0\\ 0\end{bmatrix}+e\begin{bmatrix}0\\ 1\\ 0\end{bmatrix}+h\begin{bmatrix}0\\ 0\\ 1\end{bmatrix},c\begin{bmatrix}1\\ 0\\ 0\end{bmatrix}+f\begin{bmatrix}0\\ 1\\ 0\end{bmatrix}+i\begin{bmatrix}0\\ 0\\ 1\end{bmatrix}) = abcV(\begin{bmatrix}1\\ 0\\ 0\end{bmatrix},\begin{bmatrix}1\\ 0\\ 0\end{bmatrix},\begin{bmatrix}1\\ 0\\ 0\end{bmatrix}) + abfV(\begin{bmatrix}1\\ 0\\ 0\end{bmatrix},\begin{bmatrix}1\\ 0\\ 0\end{bmatrix},\begin{bmatrix}0\\ 1\\ 0\end{bmatrix})$$

$$+ \text{abiV}(\begin{bmatrix}1\\ 0\\ 0\end{bmatrix}\text{،}\begin{bmatrix}1\\ 0\\ 0\end{bmatrix}\text{،}\begin{bmatrix}0\\ 0\\ 1\end{bmatrix}(+\ldots)allpossiblestates\ \text{:۳*۳*۳} = \text{(۲۷}+ \text{ghcV}(\begin{bmatrix}0\\ 0\\ 1\end{bmatrix}\text{،}\begin{bmatrix}0\\ 0\\ 1\end{bmatrix}\text{،}\begin{bmatrix}1\\ 0\\ 0\end{bmatrix}(+ \text{ghfV}(\begin{bmatrix}0\\ 0\\ 1\end{bmatrix}\text{،}\begin{bmatrix}0\\ 0\\ 1\end{bmatrix}\text{،}\begin{bmatrix}0\\ 1\\ 0\end{bmatrix}($$

$$+ \text{ghiV}(\begin{bmatrix}0\\ 0\\ 1\end{bmatrix}\text{،}\begin{bmatrix}0\\ 0\\ 1\end{bmatrix}\text{،}\begin{bmatrix}0\\ 0\\ 1\end{bmatrix}(= \text{aeiV}(\begin{bmatrix}1\\ 0\\ 0\end{bmatrix}\text{،}\begin{bmatrix}0\\ 1\\ 0\end{bmatrix}\text{،}\begin{bmatrix}0\\ 0\\ 1\end{bmatrix}(+ \text{afhV}(\begin{bmatrix}1\\ 0\\ 0\end{bmatrix}\text{،}\begin{bmatrix}0\\ 0\\ 1\end{bmatrix}\text{،}\begin{bmatrix}0\\ 1\\ 0\end{bmatrix}(+ \text{bdiV}(\begin{bmatrix}0\\ 1\\ 0\end{bmatrix}\text{،}\begin{bmatrix}1\\ 0\\ 0\end{bmatrix}\text{،}\begin{bmatrix}0\\ 0\\ 1\end{bmatrix}(+ \text{bfgV}(\begin{bmatrix}0\\ 0\\ 1\end{bmatrix}\text{،}\begin{bmatrix}1\\ 0\\ 0\end{bmatrix}\text{،}\begin{bmatrix}0\\ 1\\ 0\end{bmatrix}($$

$$+ \text{cdhV}(\begin{bmatrix}0\\ 1\\ 0\end{bmatrix}\text{،}\begin{bmatrix}0\\ 0\\ 1\end{bmatrix}\text{،}\begin{bmatrix}1\\ 0\\ 0\end{bmatrix}(+ \text{cegV}(\begin{bmatrix}0\\ 0\\ 1\end{bmatrix}\text{،}\begin{bmatrix}0\\ 1\\ 0\end{bmatrix}\text{،}\begin{bmatrix}1\\ 0\\ 0\end{bmatrix}(= \text{aei} - \text{afh} - \text{bdi} + \text{bfg} + \text{cdh} - \text{ceg}$$

Theorem

A square matrix A is ivertible if and only if $\det(A) \neq 0$

Theorem * : Let $n = \dim V$. Suppose α is a nonzero alternating n-linear form on V and $e_1, ..., e_n$ is a list of vectors in V. Then

$$\alpha(e_1, ..., e_n) \neq 0$$

if and only if $e_1, ..., e_n$ is linearly independent.
If A is invertible، then $AA^{-1} = I$، so

$$1 = \det(I) = \det(AA^{-1}) = \det(A)\det(A^{-1})$$

therefore $\det(A) \neq 0$.
Now suppose $\det(A) \neq 0$. Suppose $v \in V$ and $v \neq 0$. Let $v, e_2, ..., e_n$ be a basis of V and let $\alpha \in V_{alt}^{(n)}$ be such that $\alpha \neq 0$. Then $\alpha(v, e_2, ..., e_n) \neq 0$(theorem *). Now

$$\alpha(Av, Ae_2, ..., Ae_n) = (\det(A))\alpha(v, e_2, ..., e_n) \neq 0$$

Thus $Av \neq 0$. Hence A is invertible.

Theorem

If one row or column is zero، then determinant is zero.

$$A = \begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \vdots & \vdots & \vdots & \vdots \\ 0 & 0 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ a_{n1} & a_{n2} & \cdots & a_{nn} \end{bmatrix}$$

The k^{th} row is completely zero. And we know that $\det(A) = \det(A^T)$.

$$\det(A) = \sum_{j=1}^{n} (-1)^{j+1} a_{kj} \det(A_{kj})$$

Theorem

If two rows or columns of matrix are same، then determinant is zero.

The columns or rows are linearly dependent، suppose that the k^{th} column or the k^{th} row = v_k.

$$\alpha(v_1, v_2, ..., v_k, ..., v_n) = \alpha(v_1, v_2, ..., c_1v_1 + c_2v_2 + ... + c_{k-1}v_{k-1} + c_{k+1}v_{k+1} + ... + c_nv_n, ..., v_n) = 0$$

as a result the $\det(A)$ is zero، because determinant is an alternating n-linear form of F^n.

Theorem

If two rows or columns of matrix are interchanged، the sign of determinant is changes!

$A = [v_1 v_2 ... v_k ... v_p ... v_n]$ and we know that $\det(A) = \det(v_1 v_2 ... v_k ... v_p ... v_n) = \alpha(v_1 v_2 ... v_k ... v_p ... v_n)$.
Then assume that we interchange v_k and v_p. Now we build the new matrix
$B = [v_1 v_2 ... (v_k + v_p) ... (v_k + v_p) ... v_n]$.
The determinant of B is zero، because it has two linearly independent column. We have

$$0 = \alpha(v_1, ..., v_k + v_p, ..., v_k + v_p, ..., v_n)$$

$$\alpha(v_1, ..., v_p, ..., v_k, ..., v_n) = -\alpha(v_1, ..., v_k, ..., v_p, ..., v_n)$$

So the sign of determinant changed.

Theorem

$\det(I) = 1$

$$I_n = \begin{bmatrix} 1 & \cdots & \cdots & 0 \\ 0 & 1 & \cdots & \vdots \\ \vdots & \vdots & \ddots & \vdots \\ 0 & \cdots & \cdots & 1 \end{bmatrix}$$

$$\det(I_n) = det(v_1 ... v_n) = \alpha(v_1 ... v_n) = \alpha(e_1 ... e_n) = 1 \text{ Note : } e_i$$

Theorem

If a multiple of one row/column of A is added to another row/column to produce a matrix B، then $\det(A) = \det(B)$.

$A = [v_1 ... v_k ... v_p ... v_n]$ and we know that $\det(A) = \det(v_1 v_2 ... v_k ... v_p ... v_n) = \alpha(v_1 v_2 ... v_k ... v_p ... v_n)$.
Now we build matrix B، $B = [v_1 ... v_k ... (v_p + \beta v_k) ... v_n]$.

$$\det(B) = \alpha(v_1 ... v_k ... (v_p + \beta v_k) ... v_n)$$

$$= \alpha(v_1 ... v_k ... v_p ... v_n) + \alpha(v_1 ... v_k ... \beta v_k ... v_n)$$

$$= \alpha(v_1 ... v_k ... v_p ... v_n) + 0(\text{linearly dependency}) = \det(A)$$

Theorem

If A is a triangular matrix، then $\det(A)$ is the product of the entries on the main diagonal of A.

We will proof for upper triangular، recursively and note that $\det(A) = \det(A^T)$.

$$\begin{pmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ 0 & a_{22} & \cdots & a_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & a_{nn} \end{pmatrix}$$

Base case: For $n = 1$ it is trivial.

$$\det(A) = |a_{11}| = a_{11}$$

Inductive step: Assume for any $(n-1) \times (n-1)$ upper triangular matrix، the statement is hold. We can write:

$$\det(A) = \sum_{i=1}^{n} (-1)^{i+1} a_{i1} \det(A_{\backslash i \backslash 1})$$

We know that $a_{i1} = 0$ for any $i > 1$، so $\det(A) = a_{11} \det(A_{\backslash 1 \backslash 1})$ and by Induction Hypothesis $\det(A_{\backslash 1 \backslash 1}) = \Pi_{i=2}^{n} a_{ii}$. So $\det(A) = \Pi_{i=1}^{n} a_{ii}$ □

Theorem

If a column or row is multiplied to k then determinant is multiplied to k.

We will proof for multiplying a row، and for a column note that $\det(A) = \det(A^T)$.
Assume the l-th row of A is multiplied by k and yields A'.

$$\det(A') = \sum_{j=1}^{n} (-1)^{l+j} a'_{lj} \det(A'_{\backslash l \backslash j})$$

We know that $A'_{\backslash l \backslash j} = A_{\backslash l \backslash j}$ and $a'_{lj} = k a_{lj}$. So

$$\det(A') = k \sum_{j=1}^{n} (-1)^{l+j} a_{lj} \det(A_{\backslash l \backslash j}) = k \det(A)$$

Theorem

If a row/column is a multiple of another row/column then determinant is zero.

We will proof for row، and for column note that $\det(A) = \det(A^T)$.

Consider an $n \times n$ matrix A with rows $r_1, r_2, \cdots, r_n$. Suppose that row r_i is a multiple of row r_j، where $i \neq j$. That is، there exists a scalar α such that:

$$r_i = \alpha r_j$$

.

Let B be the matrix obtained by replacing row r_i with the row $r_i - \alpha r_j$. Thus:

$$r_i' = r_i - \alpha r_j = 0$$

Therefore، the new matrix B has a zero row at the i–th position)$a_{ik} = 0$ for any $0 \leq k \leq n$.(Thus:

$$\det(B) = \sum_{k=1}^{n} (-1)^{i+k} a_{ik} \det(A_{\backslash i \backslash k}) = 0$$

We will show that $\det(A) = \det(B)$. Since determinant is an alternating multilinear form:

$$\begin{aligned}\det(B) &= \det(r_1, \ldots, r_i - \alpha r_j, \ldots, r_j, \ldots, r_n) \\ &= \det(r_1, \ldots, r_i, \ldots, r_j, \ldots, r_n) \\ &\quad - \alpha \det(r_1, \ldots, r_j, \ldots, r_j, \ldots, r_n)\end{aligned}$$

The second term is zero because if swapping two identical rows negates the determinant، the determinant must be zero because it equals its own negative. The first term is $\det(A)$، so $\det(A) = \det(B) = 0$.

Theorem

If columns/rows of matrix are linearly dependent، then its determinant is zero.

We will proof for row، and for column note that $\det(A) = \det(A^T)$.

Let A with rows $r_1, \dots, r_n$ be a matrix whose rows are linearly dependent. This means there exist scalars $\alpha_1, \dots, \alpha_n$، not all zero، such that:

$$\alpha_1 r_1 + \dots + \alpha_n r_n = 0$$

Without loss of generality، we can assume r_n can be written as a linear combination of the other rows:

$$r_n = \beta_1 r_1 + \dots + \beta_{n-1} r_{n-1}$$

Where $\beta_i = -\alpha_i/\alpha_n$. So the determinant of A is:

$$\det(A) = \det(r_1, \dots, r_{n-1}, \beta_1 r_1 + \dots + \beta_{n-1} r_{n-1})$$

By the multilinear property of the determinant we can write:

$$\det(A) = \beta_1 \det(r_1, \dots, r_{n-1}, r_1) + \dots + \beta_{n-1} \det(r_1, \dots, r_{n-1}, r_{n-1})$$

By the alternating property، the determinant of a matrix with two identical rows is zero. So all terms above are equal to zero. Hence we conclude that $\det(A) = 0$.

Theorem

Columns/rows of a matrix are linearly dependent if and only if its determinant is zero.

We will proof for row، and for column note that $\det(A) = \det(A^T)$.

We will show that the determinant of matrix A is proportional to the determinant of a triangular matrix B obtained from A through row operations.

We already know that for any A we can obtain a triangular matrix B with row operations. Changes in determinant were proved previously:

Swapping rows: This change negates the determinant.

Scaling rows: Multiplying a row by k multiplies the determinant by k.

Row addition: Adding a multiple of one row to another row does not change the determinant.

So $\det(A) = \alpha \det(B)$ where $\alpha \neq 0$. Since for a matrix A with linearly independent rows، the main diagonal entities in B are all nonzero، $\det(B) = \Pi_{i=1}^{n} b_{ii} \neq 0 \Rightarrow \det(A) \neq 0$. In addition، if rows of A are linearly dependent، there exists an entity in the main diagonal equals zero، so $\det(A) = \det(B) = 0$. So، we showed that rows of a matrix are linearly dependent if and only if its determinant is zero.

Example

Compute $\det(A)$، where $A = \begin{bmatrix} 1 & -4 & 2 \\ -2 & 8 & -9 \\ -1 & 7 & 0 \end{bmatrix}$

$$\begin{vmatrix} 1 & -4 & 2 \\ -2 & 8 & -9 \\ -1 & 7 & 0 \end{vmatrix} = \begin{vmatrix} 1 & -4 & 2 \\ 0 & 0 & -5 \\ 0 & 2 & 2 \end{vmatrix} = -\begin{vmatrix} 1 & -4 & 2 \\ 0 & 3 & 2 \\ 0 & 0 & -5 \end{vmatrix} = -3\begin{vmatrix} 1 & -4 & 2 \\ 0 & 1 & \frac{2}{3} \\ 0 & 0 & -5 \end{vmatrix} = -\begin{vmatrix} 1 & 0 & \frac{14}{3} \\ 0 & 1 & \frac{2}{3} \\ 0 & 0 & -5 \end{vmatrix} = 15\begin{vmatrix} 1 & 0 & \frac{14}{3} \\ 0 & 1 & \frac{2}{3} \\ 0 & 0 & 1 \end{vmatrix} = 15\begin{vmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{vmatrix} = 15$$

Theorem

If A is an $n \times n$ matrix، then $\det(A^T) = \det(A)$

Each element $\sigma \in S_n$ has a unique inverse $\sigma^{-1} \in S_n$ such that $\sigma(\sigma_i^{-1}) = \sigma^{-1}(\sigma_i)$. We'll also need the property that $sgn(\sigma) = sgn(\sigma{-}1)$(which is clear from writing σ as a composition of interchanges and then realizing σ^{-1} is the same compsition in reverse). So، we have

$$
\begin{aligned}
\det(A) &= \sum_{\sigma \in S_n} sgn(\sigma) \prod_{i=1}^{n} a_{\sigma_i, i} \\
&= \sum_{\sigma \in S_n} sgn(\sigma) \prod_{i=1}^{n} a_{\sigma(\sigma^{-1}(i), \sigma^{-1}(i))} (\text{ reordering the product }) \\
&= \sum_{\sigma \in S_n} sgn(\sigma) \prod_{i=1}^{n} a_{i, \sigma^{-1}(i)} \\
&= \sum_{\sigma \in S_n} sgn(\sigma^{-1}) \prod_{i=1}^{n} a_{i, \sigma^{-1}(i)} \\
&= \sum_{\sigma' \in S_n} sgn(\sigma') \prod_{i=1}^{n} a_{i, \sigma'_i} (\text{writing } \sigma' \text{ for } \sigma^{-1} \text{ and reordering the sum})
\end{aligned}
$$

Theorem

If A and B are $n \times n$ matrices، then $\det(AB) = \det(A)\det(B)$

Theorem * : Let $n = \dim V$. Suppose $e_1, ..., e_n$ is a basis of V and $v_1, ..., v_n \in V$. For each $k \in \{1, ..., n\}$، let $b_{1,k}, ..., b_{n,k} \in F$ be such that

$$v_k = \sum_{j=1}^{n} b_{j,k} e_j$$

$$Then$$

$$\alpha(v_1, ..., v_n) = \alpha(e_1, ..., e_n) \sum_{(j_1,...,j_n) \in perm(n)} (sign(j_1, ..., j_n)) b_{j_1,1} ... b_{j_n,n}$$

for every alternating n–linear form α on V.

Apply theorem * with $V = F^n$ and $e_1, ..., e_n$ the standard basis of F^n and α the alternating n–linear form on F^n that takes $v_1, ..., v_n$ to $\det(v_1 ... v_n)$. If each v_k is the k^{th} column of A، then each $b_{j,k}$ in theorem * equals $A_{j,k}$. Finally،

$$\alpha(e_1, ..., e_n) = \det(e_1, , , e_n) = \det I = 1.$$

Example

Show that the determinant، $\det : \mathcal{M}_n(F) \to F$ is not a linear transformation when $n \geq 2$

It's not true، because $\det(A + B) \neq \det(A) + \det(B)$