



Determinant

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01

Introduction



Determinant of a matrix

The determinant of a 2×2 matrix $A = [a_{ij}]$ is the number: Why???

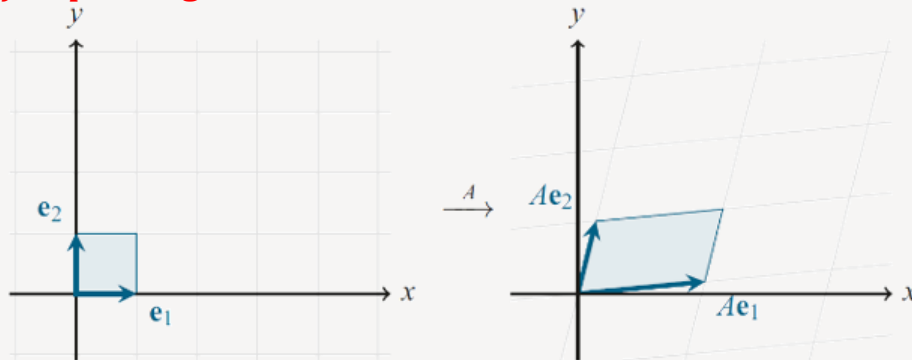
$$\det(A) = a_{11}a_{22} - a_{12}a_{21}$$

□ The absolute value of the determinant of a matrix measures how much it expands space when acting as a linear transformation. That is, it is the area (or volume, or hypervolume, depending on the dimension) of the output of the unit square, cube, or hypercube after it is acted upon by the matrix.

Geometric interpretation


□ The volume is a n -alternating multilinear map on all n -parallelpipeds such that the volume of standard unit paralleliped is one.

$$\frac{\text{volume of output region}}{\text{volume of input region}}$$



A 2×2 matrix A stretches the unit square (with sides e_1 and e_2) into a parallelogram with sides Ae_1 and Ae_2 (the columns of A). The determinant of A is the area of this parallelogram.

Geometric interpretation


$$a_2 = \begin{pmatrix} 0 \\ 1 \end{pmatrix}$$
$$a_1 = \begin{pmatrix} 1 \\ 0 \end{pmatrix}$$

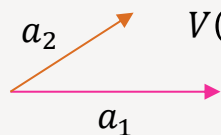
$$A = [a_1 \quad a_2], a_2 = \alpha a_1$$

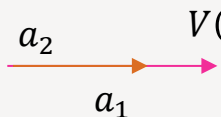
$$V(a_1, a_2) = -V(a_2, a_1)$$

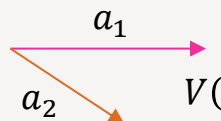

A_1

A_2

A_3


$$V(A_1) > 0$$

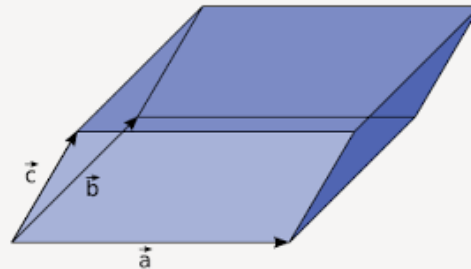
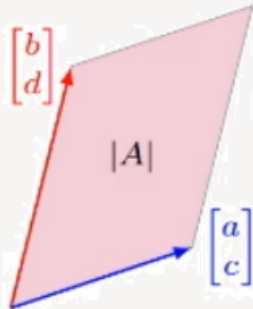

$$V(A_2) = 0$$


$$V(A_3) < 0$$


Determinants as Area or Volume

- If A is a 2×2 matrix, the **area** of the parallelogram determined by the columns of A is $\det(A)$
- If A is a 3×3 matrix, the **volume** of the parallelepiped determined by the columns of A is $\det(A)$
- Examples:

Volume of $\begin{bmatrix} \cos \theta & -\sin \theta \\ \sin \theta & \cos \theta \end{bmatrix}$ It is a rotation with θ degree



Volume

Definition

Every n -dimensional parallelepiped with $\{a_1, \dots, a_n\}$ as legs is associated with a real number, called its volume which has the following properties:

If we stretch a parallelepiped by multiplying one of its legs by a scalar λ , its volume gets multiplied by λ .

If we add a vector ω to i -th legs of a n -dimensional parallelepiped with $\{a_1, \dots, a_i, a_{i+1}, \dots, a_n\}$, then its volume is the sum of the volume from $\{a_1, \dots, a_{i-1}, a_i, a_{i+1}, \dots, a_n\}$ and the volume of $\{a_1, \dots, a_{i-1}, \omega, a_{i+1}, \dots, a_n\}$.

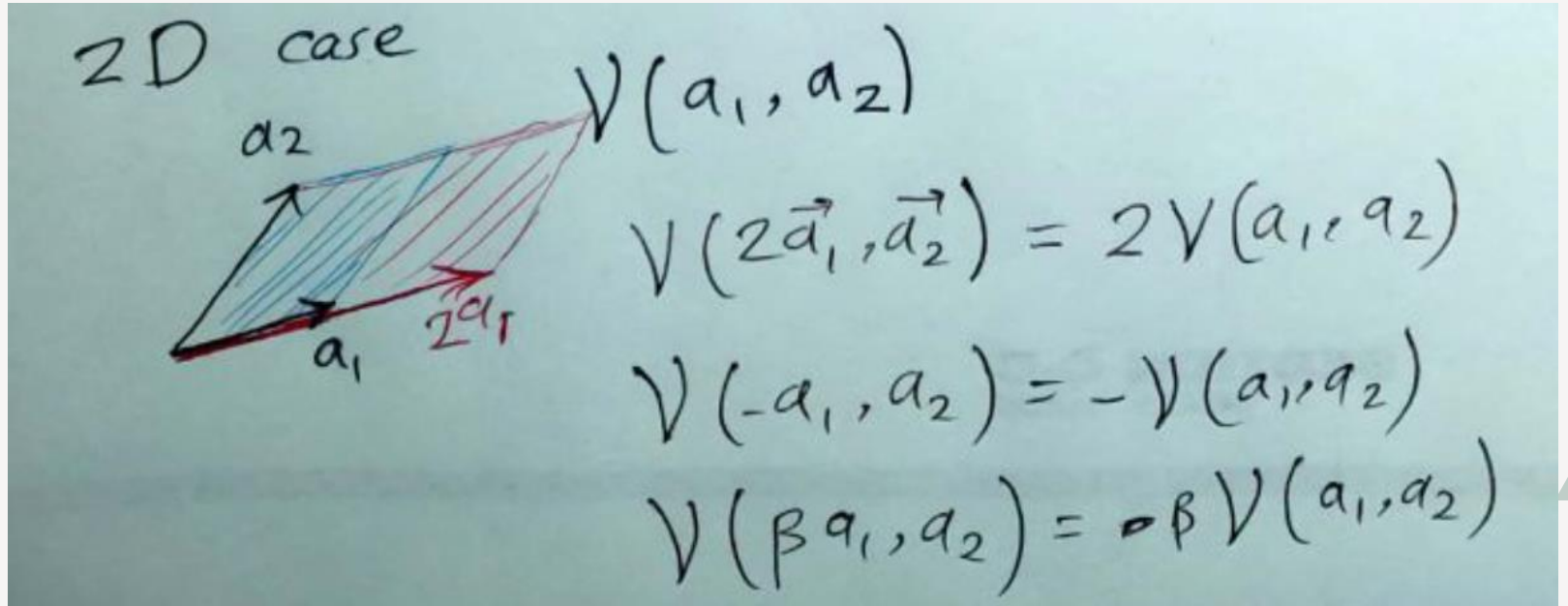
The volume changes sign when two legs are exchanged.

The volume of the parallelepiped with $\{e_1, \dots, e_n\}$ is one.

$$\phi : \underbrace{V \times \dots \times V}_n \rightarrow \mathbb{R}$$

Volume

- Example



02



Bilinear Form: Review and Continue

Bilinear Form over a complex vector

Definition

Suppose V and W are vector spaces over the same field \mathbb{C} . Then a function $\alpha: V \times W \rightarrow \mathbb{C}$ is called a **bilinear form** if it satisfies the following properties:

It is **linear in its first argument**:

$$\alpha(\mathbf{v}_1 + \mathbf{v}_2, \mathbf{w}) = \alpha(\mathbf{v}_1, \mathbf{w}) + \alpha(\mathbf{v}_2, \mathbf{w}) \text{ and}$$

$$\alpha(\lambda \mathbf{v}_1, \mathbf{w}) = \lambda \alpha(\mathbf{v}_1, \mathbf{w}) \text{ for all } \lambda \in \mathbb{C}, \mathbf{v}_1, \mathbf{v}_2 \in V, \text{ and } \mathbf{w} \in W.$$

It is **conjugate linear in its second argument**:

$$\alpha(\mathbf{v}, \mathbf{w}_1 + \mathbf{w}_2) = \alpha(\mathbf{v}, \mathbf{w}_1) + \alpha(\mathbf{v}, \mathbf{w}_2) \text{ and}$$

$$\alpha(\mathbf{v}, \lambda \mathbf{w}_1) = \bar{\lambda} \alpha(\mathbf{v}, \mathbf{w}_1) \text{ for all } \lambda \in \mathbb{C}, \mathbf{v} \in V, \text{ and } \mathbf{w}_1, \mathbf{w}_2 \in W.$$

The set of bilinear forms on \mathbf{v} is denoted by \mathbf{v}^2 .

Alternating bilinear form

Definition

A bilinear form $\alpha \in V^{(2)}$ is called alternating if

$$\alpha(v, v) = 0$$

for all $v \in V$. The set of alternating bilinear forms on V is denoted by $V_{alt}^{(2)}$.

Example

Suppose $\varphi, \tau \in V'$. Then the bilinear form α on V defined by is alternating.

$$\alpha(u, \omega) = \varphi(u)\tau(\omega) - \varphi(\omega)\tau(u)$$

Alternating bilinear form

Theorem

A bilinear form α on V is alternating if and only if

$$\alpha(u, \omega) = -\alpha(\omega, u)$$

For all $u, \omega \in V$.



Proof



Alternating bilinear form

Theorem

The sets $V_{sym}^{(2)}$ and $V_{alt}^{(2)}$ are subspaces of $V^{(2)}$. Furthermore,

$$V^{(2)} = V_{sym}^{(2)} \oplus V_{alt}^{(2)}$$



Proof

03

Multilinear Form



Multilinear Forms

Definition

Suppose $\mathcal{V}_1, \mathcal{V}_2, \dots, \mathcal{V}_p$ are vector spaces over the same field \mathbb{F} . A function

$$f : \mathcal{V}_1 \times \mathcal{V}_2 \times \dots \times \mathcal{V}_p \rightarrow \mathbb{F}$$

is called a multilinear form if, for each $1 \leq j \leq p$ and each $v_1 \in \mathcal{V}_1, v_2 \in \mathcal{V}_2, \dots, v_p \in \mathcal{V}_p$, it is the case that the function $g : \mathcal{V}_j \rightarrow \mathbb{F}$ defined by

$$g(v) = f(v_1, \dots, v_{j-1}, v, v_{j+1}, \dots, v_p) \quad \text{for all } v \in \mathcal{V}_j$$

is a linear form.

Example

Suppose $\alpha, \rho \in V^{(2)}$. Define a function $\beta: V^4 \rightarrow F$ by then $\beta \in V^4$

$$\beta(v_1, v_2, v_3, v_4) = \alpha(v_1, v_2)\rho(v_3, v_4)$$

Multilinear Forms

Definition

Suppose m is a positive integer.

- An m -linear form α on V is called *alternating* if $\alpha(v_1, \dots, v_m) = 0$ whenever v_1, \dots, v_m is a list of vectors in V with $v_j = v_k$ for some two distinct values of j and k in $\{1, \dots, m\}$.
- $V_{alt}^{(m)} = \{\alpha \in V^{(m)} : \alpha \text{ is an alternating } m\text{-linear form on } V\}$.

Theorem

$V_{alt}^{(m)}$ is a subspace of $V^{(m)}$.

Proof

Review: Characterization of Linearly Dependent sets

Theorem

An indexed set $S = \{v_1, \dots, v_n\}$ of two or more vectors is linearly dependent **if and only if** at least one of the vectors in S is a linear combination of the others. In fact, if S is linearly dependent and $v_1 \neq 0$, then **some** v_j (with $j > 1$) is a linear combination of the preceding vectors, v_1, \dots, v_{j-1} .

- ❑ Does not say that every vector
- ❑ Does not say that every vector in a linearly dependent set is a linear combination of the preceding vectors. A vector in a linearly dependent set may fail to be a linear combination of the other vectors.

Alternating multilinear forms and linear dependence

Theorem

Suppose m is a positive integer and α is an alternating m -linear form on V . If v_1, \dots, v_m is a linearly dependent list in V , then

$$\alpha(v_1, \dots, v_m) = 0$$

Proof

No nonzero alternating m -linear forms for $m > \dim V$

Theorem

Suppose $m(\text{number of vectors}) > \dim V$.

Then 0 is the only alternating m -linear form on V .



Proof



Swapping input vectors in an alternating multilinear form

Theorem

Suppose m is a positive integer, α is an alternating m -linear form on V , and v_1, \dots, v_m is a list of vectors in V . Then swapping the vectors in any slotwots of $\alpha(v_1, \dots, v_m)$ changes the value of α by a factor of -1 .

Okay, clearing up the last detail. Suppose we know that $A(e_1, e_2, e_3, e_4, e_5) = 7$. What should $A(e_3, e_5, e_1, e_2, e_4)$ be?

$$\begin{aligned} A(e_3, e_5, e_1, e_2, e_4) &= -A(e_3, e_4, e_1, e_2, e_5) \\ &= A(e_3, e_2, e_1, e_4, e_5) \\ &= -A(e_1, e_2, e_3, e_4, e_5) = -7 \end{aligned}$$

What if we did the switching in a different order? Would we get the same sign? It turns out that, yes, we would!

Permutation

Definition

Suppose m is a positive integer.

A permutation of $(1, \dots, m)$ is a list (j_1, \dots, j_m) that contains each of the numbers $1, \dots, m$ exactly once.

The set of all permutations of $(1, \dots, m)$ is denoted by $\text{perm } m$.

Example

What we need to show is that there is a way to assign a sign to

- every permutation of $\{1, 2, 3, \dots, k\}$ such that, switching the order of any two elements, switches the sign. For example:

$$(1, 2, 3) \rightsquigarrow 1 \quad (1, 3, 2) \rightsquigarrow -1$$

$$(2, 1, 3) \rightsquigarrow -1 \quad (2, 3, 1) \rightsquigarrow 1$$

$$(3, 1, 2) \rightsquigarrow 1 \quad (3, 2, 1) \rightsquigarrow -1$$

Here is the rule: The sign of $(\sigma(1), \sigma(2), \dots, \sigma(k))$ is

$$(-1)^{\#\{(i,j) : i < j \text{ and } \sigma(i) > \sigma(j)\}}.$$

$$A(e_{j_1}, e_{j_2}, \dots, e_{j_k}) = \text{sign}(\sigma) A(e_{j_{\sigma(1)}}, e_{j_{\sigma(2)}}, \dots, e_{j_{\sigma(k)}}).$$

Permutation

Definition

The *sign* of a permutation (j_1, \dots, j_m) is defined by

$$\text{sign}(j_1, \dots, j_m) = (-1)^N$$

Where N is the number of pairs of integers (k, l) with $1 \leq k < l \leq m$ such that k appears after l in the list (j_1, \dots, j_m) .

Example

The permutation $(1, \dots, m)$ [no changes in the natural order] has sign 1.

The only pair of integers (k, l) with $k < l$ such that k appears after l in the list $(2, 1, 3, 4)$ is $(1, 2)$.

Thus the permutation $(2, 1, 3, 4)$ has sign -1 .

In the permutation $(2, 3, \dots, m, 1)$, the only pairs (k, l) with $k < l$ that appear with changed order are $(1, 2), (1, 3), \dots, (1, m)$. Because we have $m - 1$ such pairs, the sign of this permutation equals $(-1)^{m-1}$.

Permutations and alternating multilinear forms

Theorem

Suppose m is a positive integer and $\alpha \in V_{alt}^{(m)}$. Then

$$\alpha(v_{j_1}, \dots, v_{j_m}) = \text{sign}(j_1, \dots, j_m) \alpha(v_1, \dots, v_m)$$

for every list v_1, \dots, v_m of vectors in V and all $(j_1, \dots, j_m) \in \text{perm } m$.

Proof

Formula for $(\dim V)$ -linear alternating

Theorem

Let $n = \dim V$. Suppose e_1, \dots, e_n is a basis of V and $v_1, \dots, v_n \in V$. For each $k \in \{1, \dots, n\}$, let $b_{1,k}, \dots, b_{n,k} \in F$ be such that

$$v_k = \sum_{j=1}^n b_{j,k} e_j$$

Then

$$v_1 = \begin{bmatrix} a \\ b \end{bmatrix}, v_2 = \begin{bmatrix} c \\ d \end{bmatrix}$$

$$\alpha(v_1, \dots, v_n) = \alpha(e_1, \dots, e_n) \sum_{(j_1, \dots, j_n) \in \text{perm}(n)} (\text{sign}(j_1, \dots, j_n)) b_{j_1, 1} \dots b_{j_n, n}$$

for every alternating n -linear form α on V .

Proof

Nonzero alternating n -linear form α on

Theorem

The vector space $\alpha_{alt}^{(\dim V)}$ with inputs from vector space V has dimension one.

Proof

Theorem

$$\alpha(v_1, \dots, v_n) = \sum_{(j_1, \dots, j_n) \in \text{perm}(n)} (\text{sign}(j_1, \dots, j_n)) \varphi_{j_1}(v_1) \dots \varphi_{j_n}(v_n)$$

The verification that α is an n -linear form on V is straightforward.

$$\alpha(e_1, \dots, e_n) = 1$$

04

Matrix Determinant



Determinant

Definition

Non-square matrices do not have determinants.

Suppose that m is a positive integer and $T \in \mathcal{L}(V)$. For $\alpha \in V_{alt}^{(m)}$, define $\alpha \in V_{alt}^{(m)}$ by

$$\alpha_T(v_1, \dots, v_m) = \alpha(Tv_1, \dots, Tv_m)$$

for each list v_1, \dots, v_m of vectors in V .

$$\alpha_T = (\det T)\alpha$$

The vector space $V_{alt}^{(\dim V)}$ has dimension one.

Example

$$\begin{aligned} V\left(\begin{bmatrix} a & c \\ b & d \end{bmatrix}\right) &= V\left(\begin{bmatrix} a \\ b \end{bmatrix}, \begin{bmatrix} c \\ d \end{bmatrix}\right) \\ &= V\left(a\begin{bmatrix} 1 \\ 0 \end{bmatrix} + b\begin{bmatrix} 0 \\ 1 \end{bmatrix}, c\begin{bmatrix} 1 \\ 0 \end{bmatrix} + d\begin{bmatrix} 0 \\ 1 \end{bmatrix}\right) \\ &= aV\left(\begin{bmatrix} 1 \\ 0 \end{bmatrix}, c\begin{bmatrix} 1 \\ 0 \end{bmatrix} + d\begin{bmatrix} 0 \\ 1 \end{bmatrix}\right) + bV\left(\begin{bmatrix} 0 \\ 1 \end{bmatrix}, c\begin{bmatrix} 1 \\ 0 \end{bmatrix} + d\begin{bmatrix} 0 \\ 1 \end{bmatrix}\right) \\ &= acV\left(\begin{bmatrix} 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 1 \\ 0 \end{bmatrix}\right) + adV\left(\begin{bmatrix} 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \end{bmatrix}\right) + bcV\left(\begin{bmatrix} 0 \\ 1 \end{bmatrix}, \begin{bmatrix} 1 \\ 0 \end{bmatrix}\right) + bdV\left(\begin{bmatrix} 0 \\ 1 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \end{bmatrix}\right) \\ &= adV\left(\begin{bmatrix} 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \end{bmatrix}\right) + bcV\left(\begin{bmatrix} 0 \\ 1 \end{bmatrix}, \begin{bmatrix} 1 \\ 0 \end{bmatrix}\right) \\ &= ad - bcV\left(\begin{bmatrix} 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \end{bmatrix}\right) = ad - bc \\ V\left(\begin{bmatrix} a & c \\ b & d \end{bmatrix}\right) &= ad - bc \\ &\text{determinant} \end{aligned}$$

Determinant

Example

Let $n = \dim V$.

- If I is the identity operator on V , then $\alpha_1 = \alpha$ for all $\alpha \in V_{alt}^{(n)}$. Thus $\det I = 1$.
- More generally, if $\lambda \in \mathbf{F}$, then $\alpha_{\lambda I} = \lambda^n \alpha$ for all $\alpha \in V_{alt}^{(n)}$. Thus $\det(\lambda I) = \lambda^n$.
- Still more generally, if $T \in \mathcal{L}(V)$ and $\lambda \in \mathbf{F}$, then $\alpha_{\lambda T} = \lambda^n \alpha_T = \lambda^n (\det T) \alpha$ for all $\alpha \in V_{alt}^{(n)}$. Thus $\det(\lambda T) = \lambda^n \det T$.

Determinant is an alternating

Theorem

Suppose that n is a positive integer. The map that takes a list v_1, \dots, v_n of vectors in \mathbf{F}^n to $\det(v_1, \dots, v_n)$ is an alternating n -linear form of \mathbf{F}^n .



Matrix Determinant

Theorem

Suppose that n is a positive integer and A is an n -by- n square matrix. Then

$$\det A = \sum_{(j_1, \dots, j_n) \in \text{perm}(n)} (\text{sign}(j_1, \dots, j_n)) A_{j_1, 1} \dots A_{j_n, n}$$

Proof

Example

Determinant of 2*2 matrix

Determinant of 3*3 matrix



Definition of Submatrix A_{ij}

Definition

For any square matrix A , let A_{ij} denote the submatrix formed by deleting the i th row and j th column of A



For instance, if

$$A = \begin{bmatrix} 1 & -2 & 5 & 0 \\ 2 & 0 & 4 & -1 \\ 3 & 1 & 0 & 7 \\ 0 & 4 & -2 & 0 \end{bmatrix}$$

A_{12} is

$$A_{12} = \begin{bmatrix} 2 & 4 & -1 \\ 3 & 0 & 7 \\ 0 & -2 & 0 \end{bmatrix}$$



Recursive Definition of Determinant

Definition

The determinant of an $n \times n$ matrix $A = [a_{ij}]$ is the sum of n terms of the form $\pm a_{1j} \det(A_{1j})$, with **plus and minus signs alternating**, where the entries $a_{11}, a_{12}, \dots, a_{1n}$ are from the first row of A . In symbols,

$$\begin{aligned} \det(A) &= a_{11} \det(A_{11}) - a_{12} \det(A_{12}) + \dots \\ &+ (-1)^{1+n} a_{1n} \det(A_{1n}) \\ &= \sum_{j=1}^n (-1)^{1+j} a_{1j} \det(A_{1j}) \end{aligned}$$

Recursive Definition of Determinant

□ 2×2 matrix

$$|A| = \sum_{j=1}^n (-1)^{i+j} a_{ij} |A_{ij}| \quad i = 1$$

$$A = \begin{bmatrix} a & b \\ c & d \end{bmatrix} \rightarrow |A| = (-1)^{1+1} a_{11} |A_{11}| + (-1)^{1+2} a_{12} |A_{12}|$$

$$\begin{vmatrix} \square & \square \\ \square & \square \end{vmatrix} = a \begin{vmatrix} \square \end{vmatrix} - b \begin{vmatrix} \square \\ c \end{vmatrix} = ad - bc$$

Example

$$\begin{vmatrix} -1 & 2 \\ -3 & 1 \end{vmatrix} = (-1) \times (1) - (2) \times (-3) = 5$$

Recursive Definition of Determinant

□ 3×3 matrix

$$|A| = \sum_{j=1}^n (-1)^{i+j} a_{ij} |A_{ij}| \quad i = 1$$

$$A = \begin{bmatrix} a & b & c \\ d & e & f \\ g & h & i \end{bmatrix} \rightarrow |A|$$

$$= (-1)^{1+1} a_{11} |A_{11}| + (-1)^{1+2} a_{12} |A_{12}| + (-1)^{1+3} a_{13} |A_{13}|$$

$$= a \begin{vmatrix} \square & \square & \square \\ \square & e & f \\ \square & h & i \end{vmatrix} - b \begin{vmatrix} \square & \square & \square \\ d & \square & f \\ g & \square & i \end{vmatrix} + c \begin{vmatrix} \square & \square & \square \\ d & e & \square \\ g & h & \square \end{vmatrix}$$

$$= a(ei - fh) - b(di - fg) + c(dh - eg)$$


$$= aei + bfg + cdh - afh - bdi - ceg$$

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Recursive Definition of Determinant

Example


$$\begin{vmatrix} 1 & 0 & 1 \\ 2 & 5 & 4 \\ 5 & 3 & -1 \end{vmatrix} = -5 + 0 + 6 - (25 + 12 + 0) = -36$$



Cofactor

Definition

Given $A = [a_{ij}]$, the (i, j) -cofactor of A is the number C_{ij} given by

$$C_{ij} = (-1)^{i+j} \det(A_{ij})$$

Then

$$\det(A) = a_{11}C_{11} + a_{12}C_{12} + \cdots + a_{1n}C_{1n}$$

Which is a cofactor expansion across the first row of A .

Cofactor Expansion

Important

The determinant of an $n \times n$ matrix A can be computed by a cofactor expansion **across any row or down any column**. The expansion across the i th row using the cofactor is

$$\det(A) = a_{i1}C_{i1} + a_{i2}C_{i2} + \cdots + a_{in}C_{in}$$

The cofactor expansion down the j th column is

$$\det(A) = a_{1j}C_{1j} + a_{2j}C_{2j} + \cdots + a_{nj}C_{nj}$$

Cofactor Expansion

Example

$$A = \begin{bmatrix} + & - & + & \dots \\ - & + & - & \dots \\ + & - & + & \dots \\ \vdots & \vdots & \vdots & \ddots \end{bmatrix} \quad A = \begin{bmatrix} 1 & 0 & 1 \\ 2 & 5 & 4 \\ 5 & 3 & -1 \end{bmatrix}$$

$$|A| = +1 \times \begin{vmatrix} 5 & 4 \\ 3 & -1 \end{vmatrix} - 0 \times \begin{vmatrix} 2 & 4 \\ 5 & -1 \end{vmatrix} + 1 \times \begin{vmatrix} 2 & 5 \\ 5 & 3 \end{vmatrix} = -36$$

$$|A| = -0 \times \begin{vmatrix} 2 & 4 \\ 5 & -1 \end{vmatrix} + 5 \times \begin{vmatrix} 1 & 1 \\ 5 & -1 \end{vmatrix} - 3 \times \begin{vmatrix} 1 & 1 \\ 2 & 4 \end{vmatrix} = -36$$

Cramer's Rule

□ $Ax = b$ and A is invertible

$$A = [A_1 \quad \dots \quad A_n] \quad I$$

$$= [e_1 \quad \dots \quad e_n]$$

$$AI = A \Rightarrow A[e_1 \quad \dots \quad e_n] = [Ae_1 \quad \dots \quad Ae_n] = [A_1 \quad \dots \quad A_n]$$

$$A \overbrace{[e_1 \quad e_2 \quad \dots \quad x \quad \dots \quad e_n]}^{I_j(x)} = [Ae_1 \quad Ae_2 \quad \dots \quad Ax \quad \dots \quad Ae_n]$$

$$= \underbrace{[A_1 \quad A_2 \quad \dots \quad b \quad \dots \quad A_n]}_{A_j(b)}$$

$$|I_2(x)| = \begin{vmatrix} 1 & x_1 & 0 \\ 0 & x_2 & 0 \\ 0 & x_3 & 1 \end{vmatrix} = x_2 \Rightarrow |I_j(x)| = x_j$$

$$AI_j(x) = A_j(b) \Rightarrow |A||I_j(x)| = |A_j(b)| \Rightarrow x_j = \frac{|A_j(b)|}{|A|}$$

Cramer's Rule

Note

□ Let A be an invertible $n \times n$ matrix. For any \mathbf{b} in \mathbb{R}^n , the unique solution \mathbf{x} of $A\mathbf{x} = \mathbf{b}$ has entries given by

□ $x_i = \frac{|A_i(\mathbf{b})|}{|A|}, \quad i = 1, 2, \dots, n$

Example

$$\begin{cases} x_1 - x_2 + 2x_3 = 1 \\ x_1 + x_2 - x_3 = 2 \\ 2x_1 - 3x_2 + x_3 = -1 \end{cases}$$

$$\Rightarrow x_2 = \frac{\begin{vmatrix} 1 & 1 & 2 \\ 1 & 2 & -1 \\ 2 & -1 & 1 \end{vmatrix}}{\begin{vmatrix} 1 & -1 & 2 \\ 1 & 1 & -1 \\ 2 & -3 & 1 \end{vmatrix}} = \frac{-12}{-3} = 4$$

A Formula for A^{-1}

The j -th column of A^{-1} is a vector x that satisfies

$$Ax = e_j$$

By Cramer's rule

$$\{(i, j) - \text{entry of } A^{-1}\} = x_i = \frac{|A_i(e_j)|}{|A|}$$

$$|A_i(e_j)| = (-1)^{i+j} |A_{ji}|$$

$$A^{-1} = \frac{1}{|A|} \begin{bmatrix} C_{11} & C_{21} & \cdots & C_{n1} \\ C_{12} & C_{22} & \cdots & C_{n2} \\ \vdots & \vdots & \ddots & \vdots \\ C_{1n} & C_{2n} & \cdots & C_{nn} \end{bmatrix}$$

The matrix of cofactors is called the **adjugate** (or **classical adjoint**) of A , denoted by

$\text{adj } A$.

$$\begin{aligned} A &= \begin{bmatrix} a & b \\ c & d \end{bmatrix} \\ [C_{ij}] &= \begin{bmatrix} d & -b \\ -c & a \end{bmatrix} \\ \Rightarrow A^{-1} &= \frac{1}{|A|} [C_{ij}]^T = \frac{1}{ad-bc} \begin{bmatrix} d & -b \\ -c & a \end{bmatrix} \end{aligned}$$

A Formula for A^{-1}

Important

Let A be an invertible $n \times n$ matrix. Then

$$A^{-1} = \frac{1}{|A|} \operatorname{adj} A$$


05

Determinant Properties





Properties

- (1) If one row or column is zero, then determinant is zero


$$\begin{vmatrix} 0 & 0 & 0 \\ a & b & c \\ d & e & f \end{vmatrix} = 0$$

- Determinant of zero matrix is...

$$\det(A) = \sum_{j=1}^n (-1)^{1+j} a_{1j} \det(A_{1j})$$
$$\det(A) = \sum_{\sigma \in S_n} \text{sgn}(\sigma) \prod_{i=1}^n a_{i\sigma(i)}$$


Properties

- (2) If two rows or columns of matrix are same, then determinant is zero.

$$A = \begin{bmatrix} 1 & -2 & 3 \\ 1 & -2 & 3 \\ 5 & 3 & -1 \end{bmatrix}$$

$$|A| = +1 \times \begin{vmatrix} -2 & 3 \\ 3 & -1 \end{vmatrix} - (-2) \times \begin{vmatrix} 1 & 3 \\ 5 & -1 \end{vmatrix} + 3 \times \begin{vmatrix} 1 & -2 \\ 5 & 3 \end{vmatrix}$$

$$|A| = -1 \times \begin{vmatrix} -2 & 3 \\ 3 & -1 \end{vmatrix} + (-2) \times \begin{vmatrix} 1 & 3 \\ 5 & -1 \end{vmatrix} - 3 \times \begin{vmatrix} 1 & -2 \\ 5 & 3 \end{vmatrix}$$

Properties

- ❑ (3) If two rows or columns of matrix are interchanged, the sign of determinant is changes!
- ❑ (4) $\det(I) = 1$



Properties

□ (5) Row and Column Operations

- If a multiple of one row/column of A is added to another row/column to produce a matrix B , then $\det(A) = \det(B)$.

Proof?

Example

$$\begin{vmatrix} 1 & -1 & 2 \\ 0 & 2 & -3 \\ 0 & 0 & -2 \end{vmatrix} = \begin{vmatrix} 1 & -1 & 2 \\ 1 & 1 & -1 \\ 1 & -1 & 0 \end{vmatrix} = \begin{vmatrix} 1 & -1 & 6 \\ 1 & 1 & 3 \\ 1 & -1 & 4 \end{vmatrix}$$

Properties

- (6) If A is a triangular matrix, then $\det(A)$ is the product of the entries on the main diagonal of A .



$$\begin{vmatrix} a & 0 & 0 \\ 0 & b & 0 \\ 0 & 0 & c \end{vmatrix} = abc$$

$$\begin{vmatrix} a & 0 & 0 \\ d & b & 0 \\ e & f & c \end{vmatrix} = abc$$


- Determinant of identity matrix is...

- U is unitary, so that $|\det(U)|=1$



Properties

- (7) If a column or row is multiply to k then determinant is multiply to k.


$$\begin{vmatrix} ka_{11} & \dots & ka_{1n} \\ \vdots & \ddots & \vdots \\ a_{n1} & \dots & a_{nn} \end{vmatrix} = k \begin{vmatrix} a_{11} & \dots & a_{1n} \\ \vdots & \ddots & \vdots \\ a_{n1} & \dots & a_{nn} \end{vmatrix}$$

- $|kA_{n \times n}| = k^n |A_{n \times n}|$


Properties

- (8) If a row/column is multiple of another row/column then determinant is
.....



Properties

- (9) If columns/rows of matrix are linear dependent then its determinant is zero

- 
- (10) If columns/rows of matrix are linear dependent if and only if its determinant is zero.



Theorem

Theorem

A square matrix A is invertible if and only if $\det(A) \neq 0$

Example

Compute $\det(A)$, where $A = \begin{bmatrix} 3 & -1 & 2 & -5 \\ 0 & 5 & -3 & -6 \\ -6 & 7 & -7 & 4 \\ -5 & -8 & 0 & 9 \end{bmatrix}$

Echelon form

Note

- ❑ Row operations
- ❑ Let A be a square matrix.
- ❑ If a multiple of one row of A is added to another row to produce a matrix B , then $\det(B) = \det(A)$
- ❑ If two rows of A are interchanged to produce B , then $\det(B) = -\det(A)$
- ❑ If one row of A is multiplied by k to produce B , then $\det(B) = k \cdot \det(A)$

Echelon form

Example

Compute $\det(A)$, where $A = \begin{bmatrix} 1 & -4 & 2 \\ -2 & 8 & -9 \\ -1 & 7 & 0 \end{bmatrix}$

Determinant of Transpose

Theorem

if A is an $n \times n$ matrix, then $\det(A^T) = \det(A)$



Multiplicative Property

Theorem

if A and B are $n \times n$ matrices, then $\det(AB) = \det(A) \det(B)$

Look at pages 27, 34

Important

In general, $\det(A + B) \neq \det(A) + \det(B)$

□ The determinant of the inverse of an invertible matrix is the inverse of the determinant

$$AA^{-1} = I \Rightarrow |AA^{-1}| = |I| = 1 \Rightarrow |A||A^{-1}| = 1 \Rightarrow |A^{-1}| = |A|^{-1}$$

□ The determinant of orthogonal matrix is ...

Transformations

Example

Show that the determinant, $\det: \mathcal{M}_n(\mathbb{F}) \rightarrow \mathbb{F}$ is not a linear transformation when $n \geq 2$



Transformations

Note

Let $T: \mathbb{R}^2 \rightarrow \mathbb{R}^2$ be the linear transformation determined by a 2×2 matrix A . If S is a parallelogram in \mathbb{R}^2 , then

$$\{\text{area of } T(S)\} = |\det A|. \{\text{area of } S\}$$

If T is determined by a 3×3 matrix A , and if S is a parallelepiped in \mathbb{R}^3 , then

$$\{\text{volume of } T(S)\} = |\det A|. \{\text{volume of } S\}$$

Reference

- ❑ Chapter 3: Linear Algebra and Its Applications, David C. Lay.
- ❑ Chapter 9: Part B and C: Linear Algebra Done Right, Sheldon Axler.

