Linear Algebra

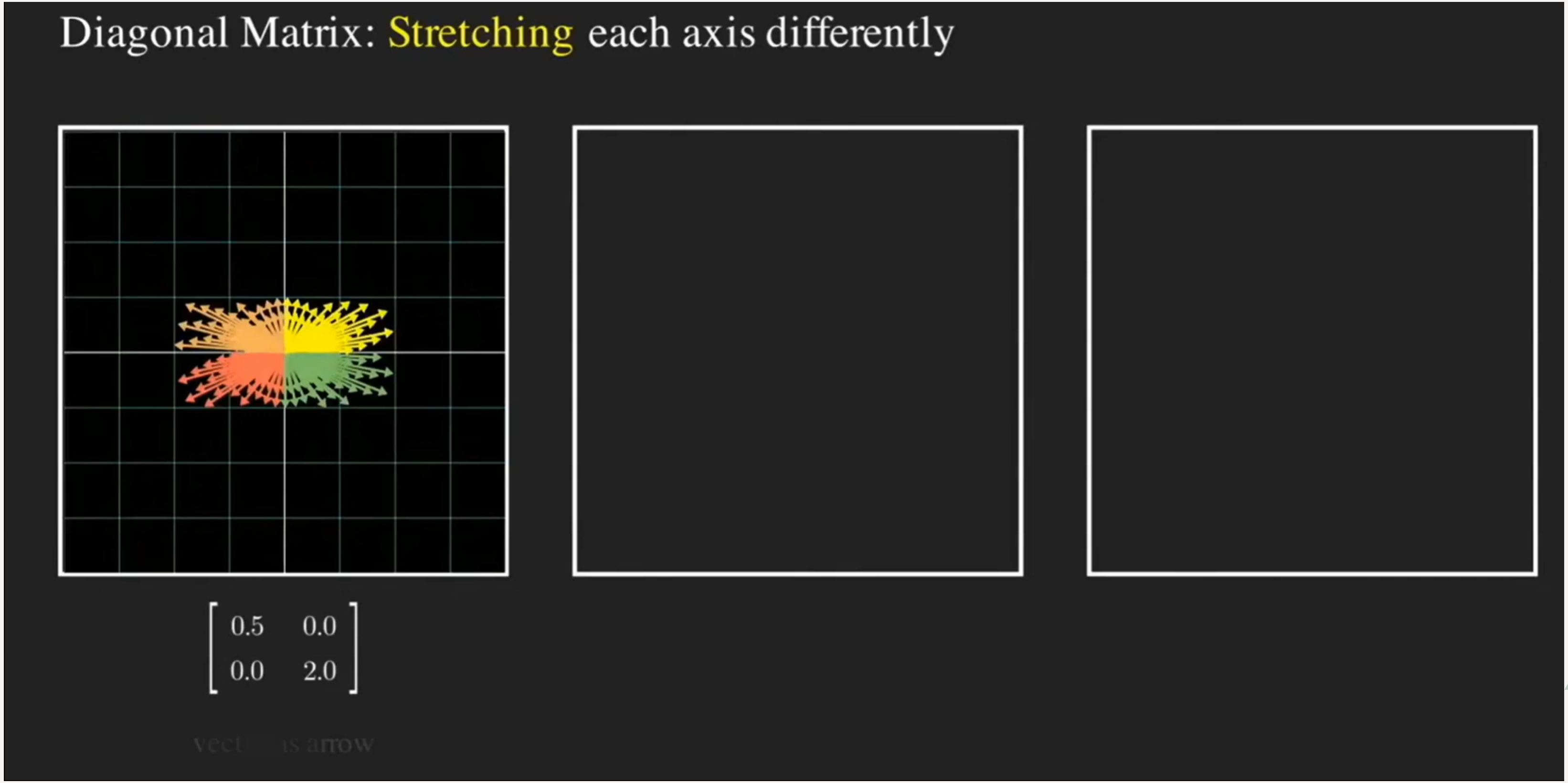
Eigenvalue – Eigenvector

Department of Computer Engineering

Sharif University of Technology

Hamid R. Rabiee rabiee@sharif.edu
Maryam Ramezani maryam.ramezani@sharif.edu

Review


Diagonal Matrix: Stretching each axis differently
$\begin{bmatrix} 0.5 & 0.0 \\ 0.0 & 2.0 \end{bmatrix}$

01

Introduction

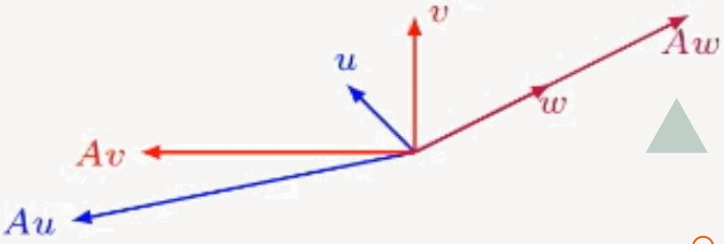
Motivation

- $A = \begin{bmatrix} 3 & -2 \\ 1 & 0 \end{bmatrix}$

  $u = \begin{bmatrix} -1 \\ 1 \end{bmatrix} \Rightarrow Au = \begin{bmatrix} 3 & -2 \\ 1 & 0 \end{bmatrix}\begin{bmatrix} -1 \\ 1 \end{bmatrix} = \begin{bmatrix} -5 \\ -1 \end{bmatrix}$

  $v = \begin{bmatrix} 0 \\ 2 \end{bmatrix} \Rightarrow Av = \begin{bmatrix} 3 & -2 \\ 1 & 0 \end{bmatrix}\begin{bmatrix} 0 \\ 2 \end{bmatrix} = \begin{bmatrix} -4 \\ 0 \end{bmatrix}$

  $w = \begin{bmatrix} 2 \\ 1 \end{bmatrix} \Rightarrow Aw = \begin{bmatrix} 3 & -2 \\ 1 & 0 \end{bmatrix}\begin{bmatrix} 2 \\ 1 \end{bmatrix} = \begin{bmatrix} 4 \\ 2 \end{bmatrix}$
- Vector “w” keeps the straight, but changes the scale.



Definition

Definition

An **eigenvector** of a square $n \times n$ matrix A is nonzero vector v such that $Av = \lambda v$ for some scalar λ. A scalar λ is called an **eigenvalue** of A if there is a nontrivial solution v of $Av = \lambda v$; such an v is called an *eigenvector corresponding to* λ.

- **An eigenvector must be nonzero, by definition, but an eigenvalue may be zero.**

Example

- $A = \begin{bmatrix} 3 & -2 \\ 1 & 0 \end{bmatrix}$, $v = \begin{bmatrix} 2 \\ 1 \end{bmatrix}$, $\lambda = 2$
- Show that 7 is an eigenvalue of matrix B, and find the corresponding eigenvectors.

$$B = \begin{bmatrix} 1 & 6 \\ 5 & 2 \end{bmatrix}$$

Eigenspace

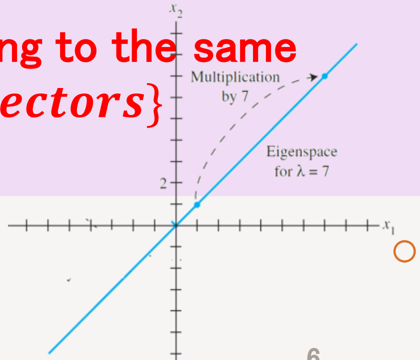
Note

λ is an eigenvalue of an $n \times n$ matrix:

$$Av = \lambda v \Rightarrow (A - \lambda I)v = 0$$

The set of all solutions of above is just the null space of the matrix $A - \lambda I$. So this set is the *subspace* of $\mathbb{R}^n$ and is called the **eigenspace** of A corresponding to λ . The eigenspace consists of the zero vector and all the eigenvectors corresponding to λ.

Eigenspace: A vector space formed by eigenvectors corresponding to the same eigenvalue and the origin point. $\boldsymbol{span\{corresponding\ eigenvectors\}}$



Definitions

Note

- $Av = \lambda v \Rightarrow Av - \lambda vI = 0 \Rightarrow (\boldsymbol{A} - \boldsymbol{\lambda I})\boldsymbol{v} = \mathbf{0} \quad \boldsymbol{v} \neq \mathbf{0}$
  - $\boldsymbol{v} \in \boldsymbol{N}(\boldsymbol{A} - \boldsymbol{\lambda I})$
  - $\boldsymbol{A} - \boldsymbol{\lambda I}$ **must be singular.**
  - Proof that for finding the eigenvalue we should solve the determinate zero equation. Look at nullspace, rank and nullity theorem, singular matrix, and det zero!
- **Characteristic polynomial** $\mathbf{det}(\boldsymbol{A} - \boldsymbol{\lambda}\mathbf{I})$
- **Characteristic equation** $\mathbf{det}(\boldsymbol{A} - \boldsymbol{\lambda}\mathbf{I}) = \mathbf{0}$
- If λ is an eigenvalue of A, then the subspace E_λ = {span{v} | Av = λ v} is called the eigenspace of A associated with λ. (This subspace contains all the span of eigenvectors with eigenvalue λ, and also the zero vector.)
- **Eigenvector is basis for eigenspace.**
- Set of all eigenvalues of matrix is $\sigma(A)$ *named spectrum of a matrix*

Definitions

Note

- Instead of $\det(A - \lambda I)$, we will compute $\mathbf{det}(\boldsymbol{\lambda}\mathbf{I} - \boldsymbol{A})$. Why?
  - $\det(A - \lambda I) = (-1)^{\mathrm{n}}\det(\lambda I - \mathrm{A})$
  - Matrix $n \times n$ with real values has ⋯⋯ eigenvalues.

Finding Eigenvalues and Eigenvectors

Let A be an $n \times n$ matrix.

1. First, find the eigenvalues λ of A by solving the equation $\det(\lambda I - A) = 0$.
2. For each λ, find the basic eigenvectors $X \neq 0$ by finding the basic solutions to $(\lambda I - A) X = 0$.

To verify your work, make sure that $AX = \lambda X$ for each λ and associated eigenvector X.

Example

Example

Find eigenvalues and eigenvectors, eigenspace (E), and *spectrum* of matrix $A = \begin{bmatrix} \mathbf{3} & \mathbf{-2} \\ \mathbf{1} & \mathbf{0} \end{bmatrix}$:

$$\det(A - \lambda \mathrm{I}) = \begin{vmatrix} 3-\lambda & -2 \\ 1 & -\lambda \end{vmatrix} = \lambda^2 - 3\lambda + 2 = 0 \Rightarrow \begin{cases} \lambda_1 = 1 \\ \lambda_2 = 2 \end{cases}$$

$$\left.\begin{array}{r} \lambda_1 = 1 \\ (A - \lambda_1 I)q_1 = 0 \end{array}\right\} \Rightarrow q_1 = \begin{bmatrix} 1 \\ 1 \end{bmatrix} \Rightarrow \begin{bmatrix} 3 & -2 \\ 1 & 0 \end{bmatrix} \begin{bmatrix} 1 \\ 1 \end{bmatrix} = 1 \begin{bmatrix} 1 \\ 1 \end{bmatrix}$$

$$\left.\begin{array}{r} \lambda_2 = 2 \\ (A - \lambda_2 I)q_2 = 0 \end{array}\right\} \Rightarrow q_2 = \begin{bmatrix} 2 \\ 1 \end{bmatrix} \Rightarrow \begin{bmatrix} 3 & -2 \\ 1 & 0 \end{bmatrix} \begin{bmatrix} 2 \\ 1 \end{bmatrix} = 2 \begin{bmatrix} 2 \\ 1 \end{bmatrix}$$

Eigenvalues={1,2}

Eigenvectors=$\{\begin{bmatrix} 1 \\ 1 \end{bmatrix}, \begin{bmatrix} 2 \\ 1 \end{bmatrix}\}$

$E_1(A) = span\{\begin{bmatrix} 1 \\ 1 \end{bmatrix}\}$ $E_2(A) = span\{\begin{bmatrix} 2 \\ 1 \end{bmatrix}\}$

$\sigma(A)$={1,2}

$$\boldsymbol{AQ = Q\Lambda} \Rightarrow \begin{bmatrix} \mathbf{3} & \mathbf{-2} \\ \mathbf{1} & \mathbf{0} \end{bmatrix} \begin{bmatrix} \mathbf{1} & \mathbf{2} \\ \mathbf{1} & \mathbf{1} \end{bmatrix} = \begin{bmatrix} \mathbf{1} & \mathbf{2} \\ \mathbf{1} & \mathbf{1} \end{bmatrix} \begin{bmatrix} \mathbf{1} & \mathbf{0} \\ \mathbf{0} & \mathbf{2} \end{bmatrix}$$

02

Eigenvalues

Expanding the Characteristic equation of A to polynomial form

Theorem

To have (1) scalar for largest degree instead of $|\boldsymbol{A} - \boldsymbol{\lambda}\mathbf{I}|$, consider $|\boldsymbol{\lambda}\mathbf{I} - \boldsymbol{A}|$

$$f(\lambda) = |\lambda I - A| = \lambda^n + c_{n-1}\lambda^{n-1} + \ldots + c_1\lambda + c_0$$ Proof?

- The n roots of this polynomial are eigenvalues!
  - $\boldsymbol{f(\lambda) = (\lambda - \lambda_1)(\lambda - \lambda_2)\ldots(\lambda - \lambda_n)}$
- What is c_{n-1}?
  - $c_{n-1} = -trace(A)$
- What is c_0?
  - $c_0 = \det(-A) = (-1)^{\mathrm{n}}\det(\mathrm{A})$

Sum and Product of eigenvalues

Theorem

If A is an n × n matrix, then the sum of the n eigenvalues of A is the trace of A.

(coefficient c_{n-1} in expanded characteristic equation)

Other view: $f(\lambda) = (\lambda - \lambda_1)(\lambda - \lambda_2) \dots (\lambda - \lambda_n)$

$$|\lambda I - A| = \lambda^n + c_{n-1}\lambda^{n-1} + \dots + c_1\lambda + c_0$$

Proof?

Theorem

If A is an n × n matrix, then the product of the n eigenvalues is the determinant of A.

(coefficient c_0 in expanded characteristic equation)

Proof?

Determinant and Eigenvalue

Theorem

$0 \in \sigma(A) \iff |A| = 0$

Proof?

Conclusion: The Invertible Matrix Theorem

Let A be an $n \times n$ matrix. Then A is invertible if and only if:

- ❑ The number 0 is not an eigenvalue of A .
- ❑ The determinant of A is not zero.

An Important Theorem!

Theorem

The eigenvalues of a triangular (upper/lower/diagonal) matrix are the entries on its main diagonal. For the diagonal matrix the eigenvectors are e_is. For upper /lower matrices, Q matrix of $\boldsymbol{AQ = Q\Lambda}$ will be upper/lower triangular matrix.

Proof?

Real Eigenvalues of different matrices

- Projection matrix
  - 0 , 1
  - If rank(P)=r with n columns, what are the repetition of the eigenvalues?
    - 0: n-r   1:r
- Reflection matrix
  - 1 , -1
- Permutation matrix
  - 1 , -1

Characteristic Equation

Example

Find the eigenvalues with their repetition and eigenvectors:

- $A = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix}$
- The characteristic polynomial of a 6×6 matrix is $\lambda^6 - 4\lambda^5 - 12\lambda^4$.
- $B = \begin{bmatrix} 3 & -2 \\ 1 & 0 \end{bmatrix}$
- $C = \begin{bmatrix} 2 & 1 \\ -1 & 0 \end{bmatrix}$
- $D = \begin{bmatrix} -2 & 1 \\ -2 & 0 \end{bmatrix}$

Eigenvalues of matrix products

Theorem

The nonzero Eigenvalues of AB equal to the nonzero eigenvalues of BA.

Why Diagonalization?

Conclusion from pervious theorems

- Theorem “The eigenvalues of a triangular (upper/lower/diagonal) matrix are the entries on its main diagonal.“ can leads to if we have matrix A and B that $D = B^{-1}AB$ be a diagonal matrix:

$$\det(\lambda I - D) = \det(\lambda I - B^{-1}AB)$$

Proof?

Similarity and Diagonalizable

Definition

Two n–by–n matrices A and B are called similar if there exists an invertible n–by–n matrix ***Q*** such that

$$A = Q^{-1}BQ$$

Definition

A matrix A is said to be diagonalizable if A is similar to a diagonal matrix D: $\mathrm{D} = Q^{-1}AQ$, that is, if $A = QDQ^{-1}$ for some invertible matrix Q and some diagonal matrix D.

03

Similarity

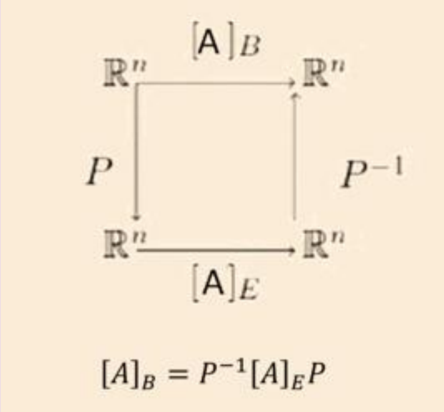
Relation between similar matrix and change of basis!

Note

- A square matrix for a linear transform

$$A: n \times n \qquad T: R^n \to R^n \quad \Rightarrow \quad \boldsymbol{Aa = b} \qquad a, b \in R^n$$

$$\left.\begin{matrix} a = P\bar{a} \\ \text{b} = P\bar{b} \end{matrix}\right\} \Rightarrow AP\bar{a} = P\bar{b} \Rightarrow \underbrace{P^{-1}AP}_{\bar{A}}\,\bar{a} = \bar{b} \Rightarrow \bar{A}\bar{a} = \bar{b}$$



- Linear transform in new basis $\bar{A} = P^{-1}AP$
- $\bar{A}$ is the standard matrix of linear transform in new basis.
- Similarity Transformation

Think!

Warnings

1. The matrices

$$\begin{bmatrix} 2 & 1 \\ 0 & 2 \end{bmatrix} \text{ and } \begin{bmatrix} 2 & 0 \\ 0 & 2 \end{bmatrix}$$

are not similar even though they have the same eigenvalues.

2. Similarity is not the same as row equivalence. (If A is row equivalent to B, then $B = EA$ for some invertible matrix E.) Row operations on a matrix usually change its eigenvalues.

- A matrix is a similarity invariant, meaning it remains unchanged under a similarity transformation.

- Why trace is a similarity invariant?
- Why rank is a similarity invariant?

Facts

Theorem

- *Similar matrices have:*
  - same determinant
  - equal characteristic equations
  - same trace
  - same rank
  - inverse of A and B are similar (if exists)

Proof?

Find matrix Q in similarity formula

Note

Two n–by–n matrices A and B are called similar if there exists an <u>invertible n–by–n matrix Q</u> such that $A = Q^{-1}BQ$. One solution for Q is the matrix whose columns are the eigenvectors of B.

Example

Find the similarity matrix of A

$$A = \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix}$$

Solution:

$$\boldsymbol{B} = \begin{bmatrix} \mathbf{1} & \mathbf{1} \\ -\boldsymbol{i} & \boldsymbol{i} \end{bmatrix}$$

04

Diagonalization

Diagonalizable

Definition

A matrix A is said to be diagonalizable if A is similar to a diagonal matrix, that is, if $A = QDQ^{-1}$ for some invertible matrix Q and some diagonal matrix D.

Theorem

An $n \times n$ matrix A is diagonalizable if and only if A has n linearly independent eigenvectors.

Corollary

- An $n \times n$ matrix with n distinct eigenvalues is diagonalizable.

Diagonalizable and Non-Diagonalizable Matrices

- Distinct eigenvalues -> eigenvectors are Linear Independent
- Duplicate eigenvalues -> 🧐 🧐

- Not all matrices are diagonalizable.
  - Example:

$$A = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}$$

- The diagonalizing matrix S is not unique.

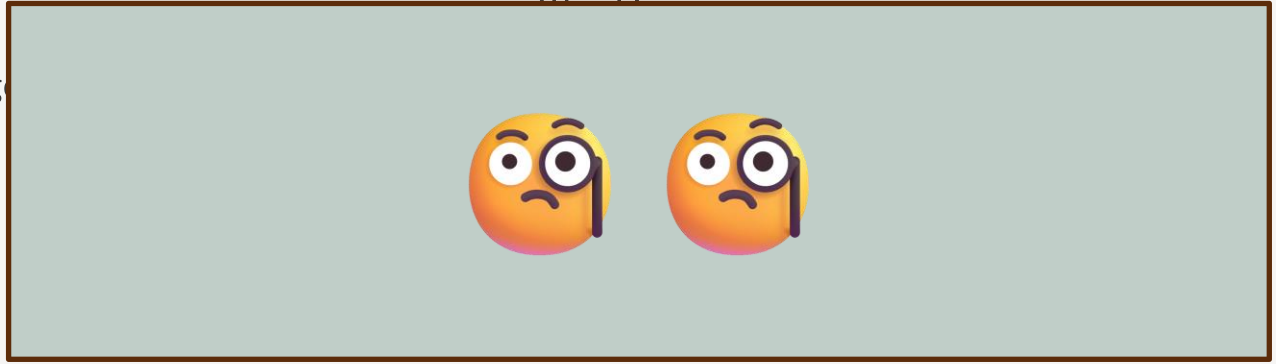
Diagonalizable and Non-Diagonalizable Matrices

- Distinct eigenvalues -> eigenvectors are Linear Independent
- Duplicate eigenvalues -> 🧐 🧐

- Not all matrices are diagonalizable.
  - Example:

[0 1]

- The diag



Diagonalizable and Non-Diagonalizable Matrices

- For matrix $A = \begin{pmatrix} 0 & -6 & -4 \\ 5 & -11 & -6 \\ -6 & 9 & 4 \end{pmatrix}$
  - Its eigenvalues are −2, −2 and −3 (repeated eigenvalues)

$$AS = SD$$

$$\begin{pmatrix} 0 & -6 & -4 \\ 5 & -11 & -6 \\ -6 & 9 & 4 \end{pmatrix}\begin{pmatrix} 0 & 0 & 2 \\ 0 & 2 & -1 \\ 0 & -3 & 3 \end{pmatrix} = \begin{pmatrix} 0 & 0 & 2 \\ 0 & 2 & -1 \\ 0 & -3 & 3 \end{pmatrix}\boxed{\begin{pmatrix} -2 & 0 & 0 \\ 0 & -2 & 0 \\ 0 & 0 & -3 \end{pmatrix}} = \begin{pmatrix} 0 & 0 & -6 \\ 0 & -4 & 3 \\ 0 & 6 & -9 \end{pmatrix}$$

Diagonal Matrix

S is not invertible!

Diagonalizable and Non-Diagonalizable Matrices

- For matrix $B = \begin{pmatrix} 4 & 8 & -2 \\ -3 & -6 & 1 \\ 9 & 12 & -5 \end{pmatrix}$
  - Its eigenvalues are −2, −2 and −3 (repeated eigenvalues)

$$BR = RD$$

$$\begin{pmatrix} 4 & 8 & -2 \\ -3 & -6 & 1 \\ 9 & 12 & -5 \end{pmatrix} \begin{pmatrix} 4 & 1 & 2 \\ -3 & 0 & -1 \\ 0 & 3 & 3 \end{pmatrix} = \begin{pmatrix} 4 & 1 & 2 \\ -3 & 0 & -1 \\ 0 & 3 & 3 \end{pmatrix} \boxed{\begin{pmatrix} -2 & 0 & 0 \\ 0 & -2 & 0 \\ 0 & 0 & -3 \end{pmatrix}} = \begin{pmatrix} -8 & -2 & -6 \\ 6 & 0 & 3 \\ 0 & -6 & -9 \end{pmatrix}$$

Diagonal Matrix

R is invertible!

Diagonalizable and Non-Diagonalizable Matrices

- For matrix $B = \begin{pmatrix} 4 & 8 & -2 \\ -3 & -6 & 1 \\ 9 & 12 & -5 \end{pmatrix}$

So what's going on here?

Diagonal Matrix

R is invertible!

Diagonalizable and Non-Diagonalizable Matrices

- Details for matrix A:

(i) For the eigenvalue -3, we have

$$\begin{pmatrix} 3 & -6 & -4 \\ 5 & -8 & -6 \\ -6 & 9 & 7 \end{pmatrix} \begin{pmatrix} X \\ Y \\ Z \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix},$$

which straightforwardly gives the eigenvector

$$\begin{pmatrix} 2 \\ -1 \\ 3 \end{pmatrix}.$$

(ii) For the repeated eigenvalue -2, we have

$$\begin{pmatrix} 2 & -6 & -4 \\ 5 & -9 & -6 \\ -6 & 9 & 6 \end{pmatrix} \begin{pmatrix} X \\ Y \\ Z \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix},$$

which equally straightforwardly gives the eigenvector

$$\begin{pmatrix} 0 \\ -2 \\ 3 \end{pmatrix}.$$

Diagonalizable and Non-Diagonalizable Matrices

(i) For the eigenvalue -3, we have

- Details for matrix B:

$$\begin{pmatrix} 7 & 8 & -2 \\ -3 & -3 & 1 \\ 9 & 12 & -2 \end{pmatrix} \begin{pmatrix} X \\ Y \\ Z \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix},$$

which, as before, straightforwardly gives the eigenvector

$$\begin{pmatrix} 2 \\ -1 \\ 3 \end{pmatrix}.$$

(ii) This time, for the repeated eigenvalue -2, we have

$$\begin{pmatrix} 6 & 8 & -2 \\ -3 & -4 & 1 \\ 9 & 12 & -3 \end{pmatrix} \begin{pmatrix} X \\ Y \\ Z \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}$$

Now, here things are different, because all three of the rows of this matrix may be reduced to the equation

$$3X + 4Y - Z = 0.$$

Diagonalizable and Non-Diagonalizable Matrices

This represents a **plane** in 3D space, and any vector in this plane is an eigenvector. We may therefore form our diagonalising matrix S out of

- Details for matrix B:

$$\begin{pmatrix} 2 \\ -1 \\ 3 \end{pmatrix}$$

together with any two non-parallel vectors of the form

$$\begin{pmatrix} X \\ Y \\ Z \end{pmatrix}$$

that satisfy

$$3X + 4Y - Z = 0;$$

that is, that are perpendicular to the vector

$$\begin{pmatrix} 3 \\ 4 \\ -1 \end{pmatrix}.$$

Both of the choices

$$S = \begin{pmatrix} 4 & 1 & 2 \\ -3 & 0 & -1 \\ 0 & 3 & 3 \end{pmatrix},$$

$$S = \begin{pmatrix} 5 & 3 & 2 \\ -3 & -3 & -1 \\ 3 & -3 & 3 \end{pmatrix}$$

will work fine, as will infinitely many others.

Diagonalizable and Non-Diagonalizable Matrices

- General considerations

  1. In general, any n by n matrix whose eigenvalues are distinct can be diagonalized.
  2. If there is a repeated eigenvalue, whether or not the matrix can be diagonalized depends on the eigenvectors.

     (i) If there k<n eigenvectors (up to multiplication by a constant), then the matrix cannot be diagonalized.

     (ii) If the unique eigenvalue corresponds to an eigenvector e, but the repeated eigenvalue corresponds to an entire plane, then the matrix can be diagonalised, using e together with any two vectors that lie in the plane.
  3. If all n eigenvalues are repeated, then things are much more straightforward: the matrix can't be diagonalized unless it's already diagonal.

Power of matrix

Example

Find $\boldsymbol{A}^{\boldsymbol{n}}$?

Conclusion

Another Notation

- With similarity transformation Q, matrix A changed to a diagonal matrix $diag(\lambda_{1,},\lambda_2)$
- Matrix A has n linear independent eigenvectors

- $[Aq_1 \;\; Aq_2 \;\; \cdots \;\; Aq_n] = \underbrace{[q_1 \;\; q_2 \;\; \cdots \;\; q_n]}_{Q} \underbrace{\begin{bmatrix} \lambda_1 & 0 & \dots & 0 \\ 0 & \lambda_2 & \dots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \dots & \lambda_n \end{bmatrix}}_{\Lambda}$

- $A\,[q_1 \, q_2 \;\; \cdots \;\; q_n] = Q\Lambda \Longrightarrow AQ = Q\Lambda$

- $\Lambda = \; Q^{-1}AQ^T$
- $A = Q\Lambda Q^{-1}$