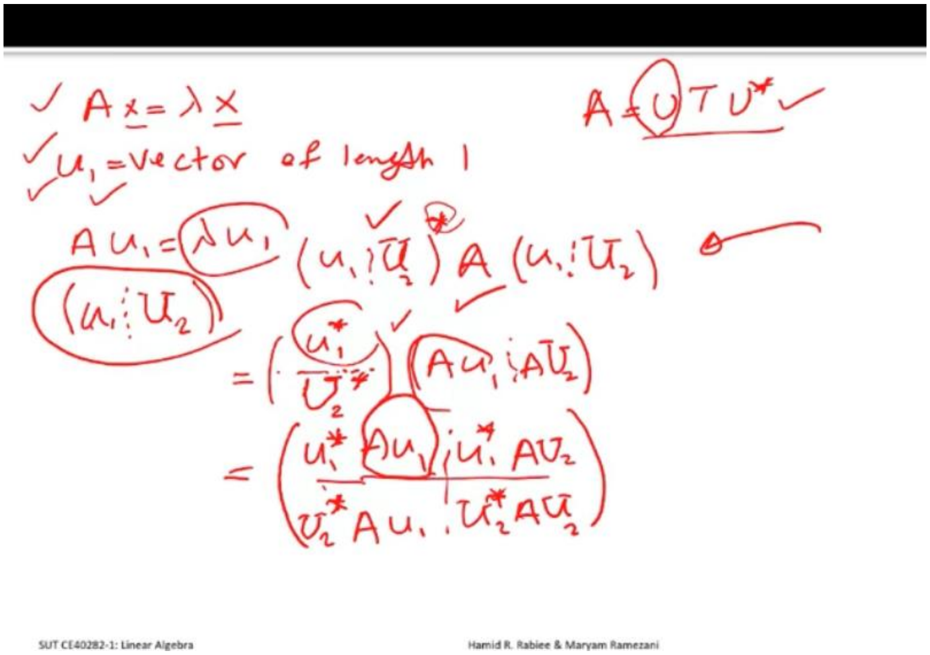
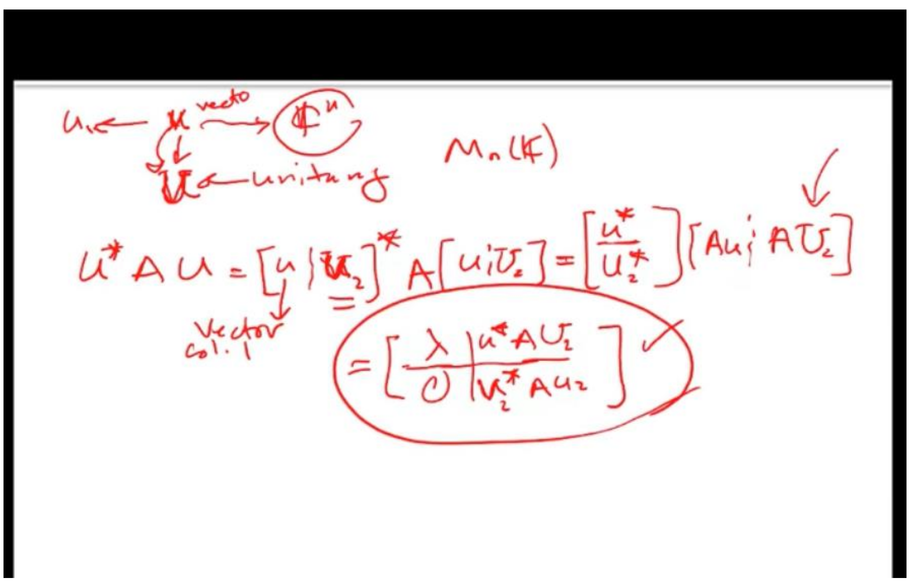
Schur Triangularization:

Proof by reduction,





$\det(A) = \lambda_1 \lambda_2 \dots \lambda_n$

proof:

$\rightarrow \det(A) = \det(UTU^*) = \det(UU^*)\det(T) = \det(T)$

$$T = \begin{bmatrix} \lambda_1 & \cdots & some\ value \\ \vdots & \ddots & \vdots \\ 0 & \cdots & \lambda_n \end{bmatrix} \rightarrow \det(T) = \lambda_1{}'\lambda_2{}' \dots \lambda_n{}'$$

$A\ and\ T\ are\ similar\ so\ they\ have\ same\ eigenvalues \rightarrow \lambda_1'\lambda_2' \dots \lambda_n' = \lambda_1\lambda_2 \dots \lambda_n \rightarrow$

$\det(A) = \lambda_1 \lambda_2 \dots \lambda_n$

$tr(A) = \lambda_1 + \cdots + \lambda_n$

proof:

$$tr(A) = tr(UTU^*) = tr(UU^*T) = tr(T) = \lambda_1 + \cdots + \lambda_n$$

Spectral decomposition:

$A = UDU^* \to A^*A = AA^*$

$\to A^*A = UD^*U^*UDU^* = UD^*DU^* = UDD^*U^* = UDU^*UD^*U^* = AA^*$

$A^*A = AA^* \to A = UDU^*$

$A = UTU^* \to UT^*U^*UTU^* = UTU^*UT^*U^* \to T^*T = TT^* \to$

$T \text{ is normal upper traingular} \to T \text{ is diagonal}$

+why?

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Denote $A = \begin{pmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ & a_{22} & \cdots & a_{2n} \\ & & \ddots & \vdots \\ & & & a_{nn} \end{pmatrix}$. Observe that the $(1, 1)$-entries of A^*A and AA^* are

$$|a_{11}|^2 \quad \text{and} \quad \sum_{i=1}^{n} |a_{1i}|^2,$$

respectively. Since A is normal,

$$\sum_{i=1}^{n} |a_{1i}|^2 = |a_{11}|^2 \quad \Rightarrow \quad \sum_{i=2}^{n} |a_{1i}|^2 = 0 \quad \Rightarrow$$
$$a_{12} = a_{13} = \cdots = a_{1n} = 0.$$

Now, from preceding result, the $(2, 2)$-entry of A^*A is

$$|a_{12}|^2 + |a_{22}|^2 = |a_{22}|^2,$$

and the $(2, 2)$-entry of AA^* is $\sum_{i=2}^{n} |a_{2i}|^2$. Again, since A is normal,

$$\sum_{i=2}^{n} |a_{2i}|^2 = |a_{22}|^2 \quad \Rightarrow \quad \sum_{i=3}^{n} |a_{2i}|^2 = 0 \quad \Rightarrow$$
$$a_{23} = a_{24} = \cdots = a_{2n} = 0.$$

Continue this process, we may conclude that the upper off-diagonal entries of A are all zero. Hence, A is a diagonal matrix.

Cholesky decomposition:

Existence:

$if\ positive\ definite\ then\ A = R^*R\ :$

$A\ is\ positive\ definite \rightarrow A\ is\ hermitian \rightarrow A\ is\ normal \rightarrow A = UDU^*$

$A\ is\ positive\ definite \rightarrow diagonal\ elements\ of\ D\ are\ Real\ and\ positive \rightarrow D = D^{\frac{1}{2}}D^{\frac{1}{2}}$

$\rightarrow A = \left(D^{\frac{1}{2}}U\right)^* D^{\frac{1}{2}}U$

$U\ is\ unitary\ and\ full\ rank, D\ is\ diagonal\ with\ positive\ elements\ so\ it'sfull\ rank$

$product\ of\ two\ full\ rank\ matrices\ is\ full\ rank.$

$\rightarrow D^{\frac{1}{2}}U = QR \rightarrow A = (QR)^*QR = R^*R$

$if\ A = R^*R\ then\ positive\ definite:$

$x^*R^*Rx = (RX)^*Rx = \left||RX|\right|^2 > 0$

Uniqueness:

$$A = MM^*$$

Then, we have

$$LL^* = MM^*$$

and

$$M^{-1}L = M^*(L^*)^{-1}$$

where the existence of the inverses $(L^*)^{-1}$ and M^{-1} is guaranteed by the fact that L and M are triangular with strictly positive diagonal entries. Since M and L are lower triangular, $M^{-1}L$ is lower triangular. Since M^* and L^* are upper triangular, $M^*(L^*)^{-1}$ is upper triangular. The lower triangular matrix $M^{-1}L$ can be equal to the upper triangular matrix $M^*(L^*)^{-1}$ only if both matrices are diagonal. Therefore,

$$M^{-1}L = D = M^*(L^*)^{-1}$$

where D is a diagonal matrix. Note that

$$\begin{aligned}(D^*)^{-1} &= \left(\left(M^*(L^*)^{-1}\right)^*\right)^{-1} \\ &= \left(L^{-1}M\right)^{-1} \\ &= M^{-1}L = D\end{aligned}$$

As a consequence,

$$DD^* = (D^*)^{-1}D^* = I$$

Thus, any diagonal entry of D (denote it by D_{kk}) satisfies

$$D_{kk}\overline{D_{kk}} = |D_{kk}|^2 = 1$$

In other words, the diagonal entries of D are all located on the unit circle. Moreover, they need to satisfy the constraint

$$L = MD$$

where the diagonal entries of both M and L are real and strictly positive. The only way to satisfy this constraint by remaining on the unit circle is to pick

$$D_{kk} = 1$$

for all k. Therefore,

$$D = I$$

and

$$L = M$$