Linear Algebra

Subspace

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Subspace Definition

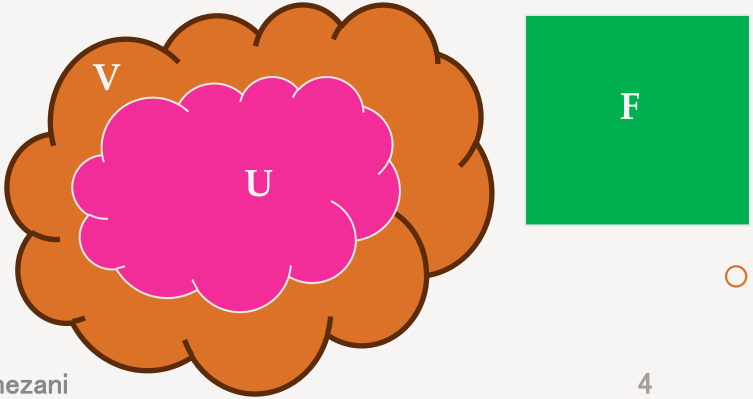
Subspace

- Zero vector is a subspace of every vector space.
- Vector space is a subspace of itself.

Definition

A **non-empty subset** of vector space for which closure holds for addition and scalar multiplication is called a subspace.

Subspace: If V is a vector space and subset $U \subseteq V$, then U is itself a vector space with the **same addition and scalar multiplication as V**.



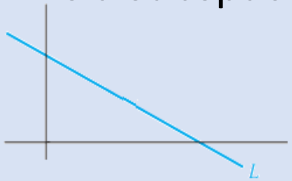
Subspace

A subspace of $\mathbb{R}^n$ is any set H in $\mathbb{R}^n$ that has these properties:

- The zero vector is in H.
- For each u and v in H, the sum u + v is in H.
- For each u in H and each scalar c, the vector cu is in H.

Example

- H = Span $\{x_1, x_2\}$, then H is a subspace of $\mathbb{R}^2$.
- Is L subspace of $\mathbb{R}^2$?



- The vector space $\mathbb{R}^2$ is a subspace of $\mathbb{R}^3$?
- Is H a subspace of $\mathbb{R}^3$? $H = \{ \begin{bmatrix} s \\ t \\ 0 \end{bmatrix} : s \text{ and } t \text{ are real}\}$

Vector Space vs Subspace

A subspace of $\mathbb{R}^n$ is any set H in $\mathbb{R}^n$ that has these properties:

- The zero vector is in H.
- For each u and v in H, the sum u + v is in H.
- For each u in H and each scalar c, the vector cu is in H.

Let V be a vector subspace and let $U \subseteq V$:

Vector Space

1. $u + v \in V$
2. $u + v = v + u$
3. $(u + v) + w = u + (v + w)$
4. There is a vector $0 \in V$ such that $u + 0 = u$
5. For each $u \in V$, there is a vector $-u \in V$ such that $u + (-u) = 0$
6. $cu \in V$
7. $c\,(u + v) = cu + cv$
8. $(c + d)u = cu + du$
9. $c\,(du) = (cd)u$
10. $1u = u$

Subspace

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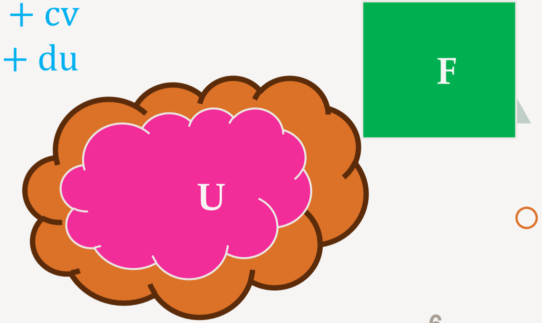


Subspace

Theorem

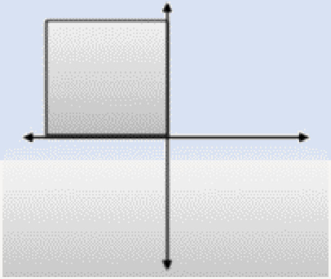
A non-empty subset U of V is a subspace of V if and only if for each pair of vectors b, c in U and each scalar α in F the vector $\alpha b + c$ is again in U.

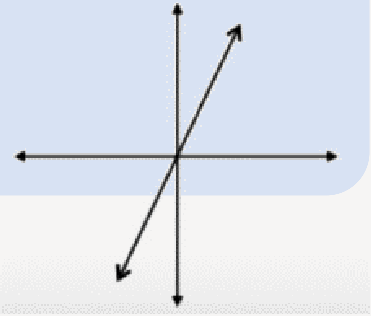
Proof:

Subspace

Example

- In F^n, the set of n-tuples $\begin{bmatrix} x_1 \\ \cdots \\ x_n \end{bmatrix}$ with $x_1 = 0$
- In F^n, the set of n-tuples $\begin{bmatrix} x_1 \\ \cdots \\ x_n \end{bmatrix}$ with $x_1 = 1 + x_2 \ (n \geq 2)$
- Every vector space with more than one member has at least _____ subspaces.
- Name subspace for $\mathbb{R}^2, \mathbb{R}^3, \mathbb{R}^4$
- Following figures are subspace of $\mathbb{R}^2$?





Subspace

Example

Let H be the set of all vectors of the form $\begin{bmatrix} a-3b \\ b-a \\ a \\ b \end{bmatrix}$ where a,b are arbitrary scalars. That is, let$H = \left\{ \begin{bmatrix} a-3b \\ b-a \\ a \\ b \end{bmatrix} : a, b \ in \ R \right\}$. Show that H is a subspace of $\mathbb{R}^4$.

Subspace

Example

- Set of all continuous real-valued functions on $\mathbb{R}$ is a subspace of the vector space of all functions on $\mathbb{R}$.
- Set of all differentiable real-valued functions on $\mathbb{R}$ is a subspace of the vector space of all functions on $\mathbb{R}$.
- Set of all functions $D(f(x)) = f'(x)$ is a subspace of the vector space of all functions on $\mathbb{R}$.

02

Intersection of subspaces

Intersection of subspaces

Theorem

If W_1 and W_2 are subspaces of V, then $W_1 \cap W_2$ is a subspace.

Proof:

$W_1 \cap W_2$ **is the largest subspace contained in** W_1 and W_2 **both.**

Intersection of subspaces

Theorem

Intersection of any collection of subspaces of a vector space V, is a subspace of V.

Proof:

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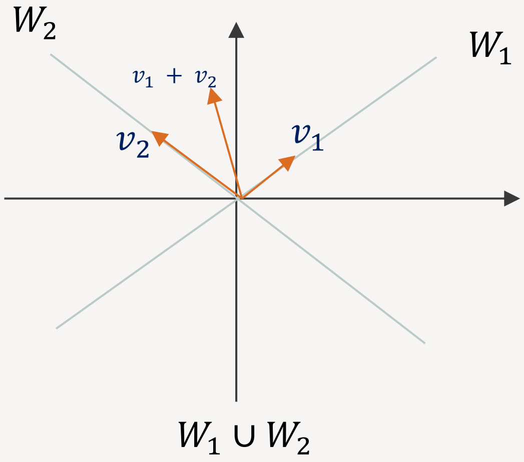
Union of subspaces

Union

Theorem

The union of two sub-spaces may not a subspace.

Proof:



Union

Theorem

Fact: The union of two sub-spaces is not a subspace unless one is contained in the other.

W_1 and W_2 are subspaces of V, then $W_1 \cup W_2$ is subspace of V if and only if $W_1 \subseteq W_2$ or $W_2 \subseteq W_1$

Proof:

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Span and Subspace

Span and Subspace

Theorem

If $v_1, v_2, \dots, v_p$ are in a vector space V, then Span $\{v_1, v_2, \dots, v_p\}$ is a subspace of V.

Proof:

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Sum of subspaces

Sum of vector spaces/subspaces

- There are two reasons to use the sum of two vector spaces.
  - to build new vector spaces from old ones.
  - to decompose the known vector space into sum of two (smaller) spaces.

- Since we consider linear transformations between vector spaces, these sums lead to representations of these linear maps and corresponding matrices into forms that reflect these sums. In many very important situations, we start with a vector space V and can identify subspaces “internally” from which the whole space V can be built up using the construction of sums.

Linear Sum of subspaces

Definition

Let A and B be non-empty subsets of a vector space V. The sum of A and B, denoted A+B, is the set of all possible sums of elements from both subsets:

$$A + B = \{a + b : a \in A, b \in B\}$$

Example

- $A = \{t_1(2,3) | t_1 is\ scalar\}$ $B = \{t_2(3,1) | t_2\ is\ scalar\}$, A+B?

- $A = \{t_1(1,2,0) | t_1 is\ scalar\}$ $B = \{t_2(0,1,2) | t_2\ is\ scalar\}$, A+B?

Linear Sum of subspaces

Theorem

If $W_1, \ldots, W_m$ are subspaces of V, then $W_1 + \cdots + W_m$ is a subspace of V.

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Direct sum of subspaces

Direct sum

Definition

U + W is called a **direct sum**, if any element in U + W can be written uniquely as u + w where $u \in U \; and \; w \in W$ (Notation: $U \oplus W$)

Example

Check where sum of following elements is a direct sum?

a) $\mathrm{U} = \begin{bmatrix} a \\ b \\ 0 \end{bmatrix}, \mathrm{W} = \begin{bmatrix} 0 \\ 0 \\ c \end{bmatrix}$

b) $\mathrm{U} = \begin{bmatrix} a \\ b \\ 0 \end{bmatrix}, \mathrm{W} = \begin{bmatrix} 0 \\ c \\ d \end{bmatrix}$

Direct Sum

Theorem

If U and W are subspaces of V, then the sum is a direct sum $U \oplus W$, if and only if

$$U \cap W = \{0\}$$

Proof:

Direct Sum

Example

Let E denote the set of all polynomials of even powers.
$E = \{a_n t^{2n} + a_{n-1} t^{2n-2} + \ \dots + a_0\}$, and O be the set of all polynomials of odd powers :
$O = \{a_n t^{2n+1} + a_{n-1} t^{2n-1} + \ \dots + a_0\}$.
The set of all polynomials P is a direct sum of E and O :

$$P = E \oplus O$$

It is easy to see that any polynomial (or function) can be uniquely decomposed into direct sum of its even and odd counterparts:

$$p(t) = \frac{p(t) + p(-t)}{2} + \frac{p(t) - p(-t)}{2}$$

Direct Sum

Example

Prove set of all bound functions such as

$$W = \{f(x) \mid \exists M \in R \; such \; that \; |f(x)| \leq M, \forall x \in R\}$$

is a subspace of $V = \{all \; functions \; from \; R \; to \; R\}$

Note

Triangle Inequality for Real Numbers

$$|a + b| \leq |a| + |b|$$

Resources

- LINEAR ALGEBRA: Theory, Intuition, Code, David Cherney.
- Chapter 4 of Elementary Linear Algebra with Applications
- Chapter 3 of Applied Linear Algebra and Matrix Analysis