Linear Algebra

Bases & Dimension

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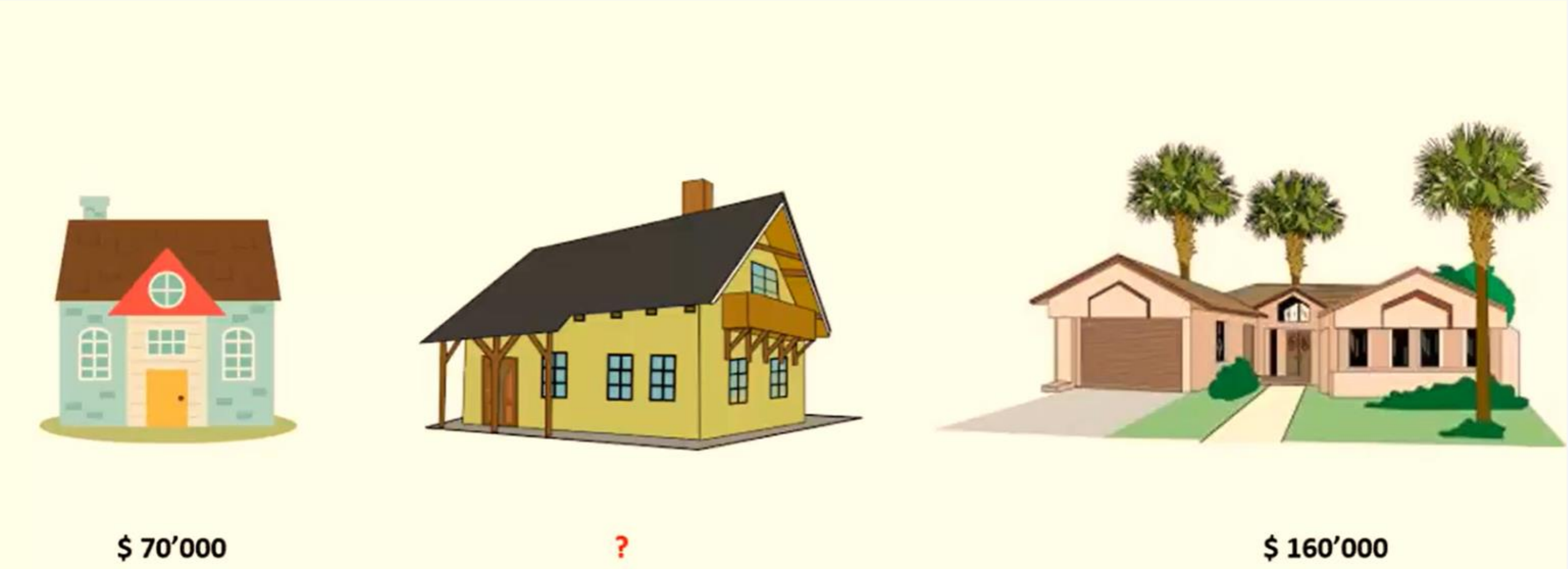
Introduction

Price Problem



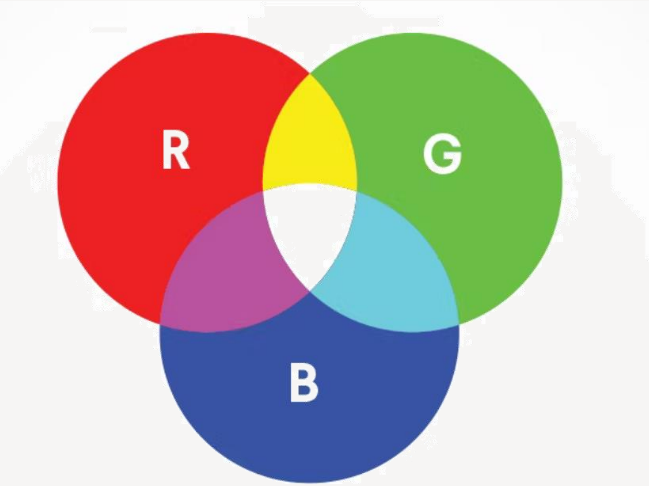
Introduction

$$\begin{bmatrix} \#Room & Size\_part1 & Size\_part2 & Size\_part3 & Size\_part4 & Size & Age & Floor & Is\_near\_park \\ & & & & & & & & \\ & & & & & & & & \end{bmatrix}$$

Home #1
Home #2
Home #3

Home #N

Introduction



02

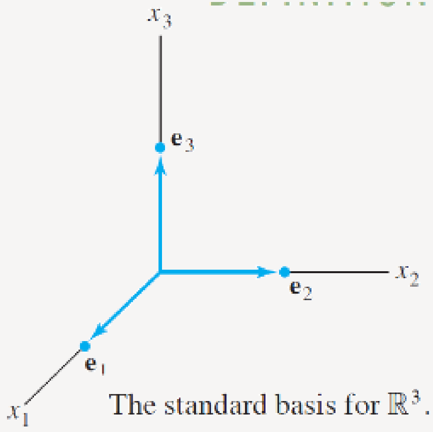
Basis

Basis

- A set of n linearly independent n-vectors is called a basis.
- A basis is the combination of span and independence: A set of vectors $\{v_1, \dots, v_n\}$ forms a basis for some subspace of $\mathbb{R}^n$ if it

  (1) spans that subspace

  (2) is an independent set of vectors.



The standard basis for $\mathbb{R}^3$.

Basis

Definition

Let H be a subspace of a vector space V. An indexed set of vectors $\mathcal{B} = \{b_1, \dots, b_n\}$ in V is a **basis** for H if

1. $\mathcal{B}$ is linearly independent set, and
2. The subspace spanned by $\mathcal{B}$ coincides with H; that is,

$$H = Span\,\{b_1, \dots, b_n\}$$

Example

Which are unique?

- ❑ Express a vector in terms of any particular basis
- ❑ Bases for $\mathbb{R}^2$
- ❑ Bases with unit length for $\mathbb{R}^2$

Vector Space of Polynomials

Be careful: A vector space can have many bases that look very different from each other!

Example (Basis)

- ❑ Standard bases for $P_n(\mathbb{R})$?
- ❑ Are $(1 - x), (1 + x), x^2$ basis for $P_2(\mathbb{R})$?

03

Dimension

Dimensions

- The dimensionality of a vector is the number of coordinate axes in which that vector exists.
- If a vector space is spanned by a finite number of vectors, it is said to be finite-dimensional. Otherwise it is infinite-dimensional.
- The number of vectors in a basis for a finite-dimensional vector space V is called the dimension of V and denoted dim(V).

Bases and finite dimension

Theorem 1

Let V be a vector space which is spanned by a finite independent set of vectors $x_1, x_2, \dots, x_m$. Then any independent set of vectors in V is finite and contains no more than m elements.

Conclusion

Every basis of V is finite and contains no more than m elements.

Independent ≤ spanning

Conclusion

In a finite-dimensional space,

$$\begin{matrix} \textit{the length of every linearly} \\ \textit{independent list of vectors} \end{matrix} \leq \begin{matrix} \textit{the length of every} \\ \textit{spanning list of vectors} \end{matrix}$$

Bases and finite dimension

Theorem 2

If V is a finite-dimensional vector space, then any two bases of V has the same (finite) number of elements.

Basis and finite dimension

The number of vectors in a basis for a finite-dimensional vector space V is called the dimension of V and denoted as dim(V).

Theorem 2

⬇

Theorem 3

Let V be a vector space with a basis B of size m. Then

a) Any set of more than m vectors in V must be linearly dependent, and

b) Any set of fewer than m vectors cannot span V.

Dimensions

Definition

A vector space V is called...

a) **finite-dimensional** if it has a finite basis, and its **dimension**, denoted by $\dim(V)$, is the number of vectors in one of its bases.

b) **infinite-dimensional** if it has no finite basis, and we say that $\dim(V) = \infty$.

Note

Dimension of subspace $\{\mathbf{0}\}$?

Dimensions

Example

Let's compute the dimension of some vector spaces that we've been working with.

Vector space	Basis	Dimension
$\mathcal{F}^n$ (n–tuples each elements from field $\mathcal{F}$)		
P^p (polynomials with max degree p)		
$M_{m,n}$ (matrices with m rows and n columns)		
P (all polynomials)		
$F\ (all\ functions)$		
C (all continues functions)		

Note!

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Finite Dimensional Subspace

Basis of Subspace

Theorem 4

If W is a subspace of a finite-dimensional vector space V, every linearly independent subset of W is finite and is part of a (finite) basis for W.

Theorem (Lemma) 5

Let S be a linearly independent subset of a vector space V. Suppose u is a vector in V which is not in the subspace spanned by S. Then the set obtained by adjoining u to S is linearly independent.

Basis of Subspace

Corollary

A subspace is called a proper subspace **if it's not the entire space**, so R2 is the only subspace of R2 which is not a proper subspace

If W is a proper subspace of a finite-dimensional vector space V, then W is finite-dimensional and $\dim(W) < \dim(V)$

Corollary

In a finite-dimensional vector space V, every non-empty linearly independent set of vectors is part of basis.

Basis of sum of subspaces

Theorem 6

If W_1 and W_2 are finite-dimensional subspaces of a vector space V, the $W_1 + W_2$ is a finite-dimensional and

$$\dim(W_1) + \dim(W_2) = \dim(W_1 \cap W_2) + \dim(W_1 + W_2)$$

Basis of sum of subspaces

Theorem 7

If $W_1,\ W_2\ and\ W_3$ are finite-dimensional subspaces of a vector space V, then can we have the following relation?

$$\dim(W_1 + W_2 + W_3)$$
$$= \dim(W_1) + \dim(W_2) + \dim(W_3) - \dim(W_1 \cap W_2)$$
$$- \dim(W_2 \cap W_3) - \dim(W_1 \cap W_3) + \dim(W_1 \cap W_2 \cap W_3)$$

Counterexample: $W_1 = span\left\{\begin{bmatrix}1\\0\end{bmatrix}\right\}, W_2 = span\left\{\begin{bmatrix}0\\1\end{bmatrix}\right\}, W_3 = span\left\{\begin{bmatrix}1\\1\end{bmatrix}\right\}$

Basis of sum of subspaces

Theorem 8

If $W_1,\ W_2\ and\ W_3$ are finite-dimensional subspaces of a vector space V, then:

$$\dim(W_1 + W_2 + W_3)$$
$$\leq \dim(W_1) + \dim(W_2) + \dim(W_3) - \dim(W_1 \cap W_2)$$
$$- \dim(W_2 \cap W_3) - \dim(W_1 \cap W_3) + \dim(W_1 \cap W_2 \cap W_3)$$

Which vector spaces have bases?

Theorem 7

Let V be a finite dimensional vector space and let W be a subspace of V. Then W has a finite basis.

Theorem 8

Let V be a vector space which has a finite spanning set. Then V has a finite basis.

Dimensionality and Properties of Bases

Note

Let V be a finite dimensional vector space over field F. Below are some properties of bases:

1. Any linearly independent list can be extended to a basis (a maximal linearly independent list is spanning).
2. Any spanning list contains a basis (a minimal spanning list is linearly independent).
3. Any linearly independent list of length $\dim V$ is a basis.
4. Any spanning list of length $\dim V$ is a basis.

We will learn about change of basis later.

05

Coordinates

Ordered basis

Definition

If V is a finite-dimensional vector space, as ordered basis for V is a finite sequence of vectors which is linearly independent and spaces V.

Be careful: The order in which the basis vectors appear in B affects the order of the entries in the coordinate vector. This is kind of janky (technically, sets don't care about order), but everyone just sort of accepts it.

Coordinate Systems

- The main reason for selecting a basis for a subspace H; instead of merely a spanning set, is that each vector in H can be written in only one way as a linear combination of the basis vectors.

Note

Suppose the set $\mathcal{B} = \{\boldsymbol{b_1}, \ldots, \boldsymbol{b_P}\}$ is a basis for a subspace H. For each x in H, the **coordinates of $\boldsymbol{x}$ relative to the basis $\boldsymbol{\mathcal{B}}$** are the weights $c_1, \ldots, c_P$ such that $\mathrm{x} = c_1 b_1 + \cdots + c_P b_p$, and the vector in $\mathbb{R}^p$

$$[x]_{\mathcal{B}} = \begin{bmatrix} c_1 \\ \vdots \\ c_P \end{bmatrix}$$

is called the **coordinate vector of $\boldsymbol{x}$ (relative to $\boldsymbol{\mathcal{B}}$)** or the $\mathcal{B}$-coordinate vector of x.

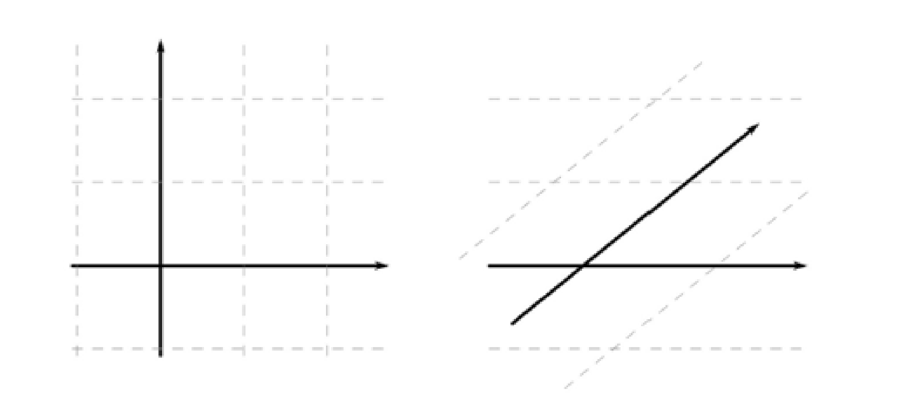
Coordinate Systems

Example

Coordinate vector of $p(x) = 4 - x + 3x^2$ respect to basis $\{1, x, x^2\}$

Coordinate axes

- The familiar Cartesian plane (left) has orthogonal coordinate axes. However, axes in linear algebra are not constrained to be orthogonal (right), and non-orthogonal axes can be advantageous.



Barycentric Coordinates

Theorem 9

Let set $S = \{v_1, \dots, v_k\}$ be an affinely independent set in $\mathbb{R}^n$. Then each $\mathbf{p}$ in aff S has a unique representation as an affine combination of $v_1, \dots, v_k$. That is, for each $\mathbf{p}$ there exists a unique set of scalers $c_1, \dots, c_k$ such that

$$\mathbf{p} = c_1 v_1 + \cdots + c_k v_k \quad and \quad c_1 + \cdots + c_k = 1$$

Note

$$\begin{bmatrix} \mathbf{p} \\ 1 \end{bmatrix} = c_1 \begin{bmatrix} v_1 \\ 1 \end{bmatrix} + \cdots + c_k \begin{bmatrix} v_k \\ 1 \end{bmatrix}$$

Involving the homogeneous forms of the points. Row reduction of the augmented matrix $[\widetilde{v_1} \ \dots \ \widetilde{v_k} \ \ \tilde{\mathbf{p}}]$ produces the Barycentric coordinates of $\mathbf{p}$.

Barycentric Coordinates

Definition

Let set $S = \{v_1, \ldots, v_k\}$ be an affinely independent set. Then for each point $\mathbf{p}$ in aff S, the coefficients $c_1, \ldots, c_k$ in the unique representation

$$\mathbf{p} = c_1 v_1 + \cdots + c_k v_k \quad and \quad c_1 + \cdots + c_k = 1$$

of $\mathbf{p}$ are called the **Barycentric** (or, sometimes **affine**) **coordinates** of $\mathbf{p}$

Barycentric Coordinates

Example

Let $a = \begin{bmatrix}1\\7\end{bmatrix}, b = \begin{bmatrix}3\\0\end{bmatrix}, c = \begin{bmatrix}9\\3\end{bmatrix}$, and $p = \begin{bmatrix}5\\3\end{bmatrix}$. Find the Barycentric Coordinates of p determined by the affinely independent set $\{a, b, c\}$.

Note

$S = \{v_1, \dots, v_k\}$ are affinely independent, if & only if $\begin{bmatrix}v_1\\1\end{bmatrix} \dots \begin{bmatrix}v_k\\1\end{bmatrix}$ are linear independent.

Resources

- Page 97 LINEAR ALGEBRA: Theory, Intuition, Code
- Page 213: David Cherney,
- Page 54: Linear Algebra and Optimization for Machine Learning